

5-dimensional geometries II: the non-fibered geometries

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May 25, 2016

Abstract

We classify the 5-dimensional homogeneous geometries in the sense of Thurston. The present paper (part 2 of 3) classifies those in which the linear isotropy representation is either irreducible or trivial. The 5-dimensional geometries with irreducible isotropy are the irreducible Riemannian symmetric spaces, while those with trivial isotropy are simply-connected solvable Lie groups of the form $\mathbb{R}^3 \rtimes \mathbb{R}^2$ or $N \rtimes \mathbb{R}$ where N is nilpotent.

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1 Introduction

Riemannian homogeneous spaces appeared in Thurston's Geometrization Conjecture as local models for pieces in a decomposition of 3-manifolds. Those with compact quotients and maximal symmetry, the eight *geometries* (see Defn. 2.1 for details), are classified by Thurston in [Thu97, Thm. 3.8.4].

Filipkiewicz classified the 4-dimensional geometries in [Fil83], retaining the conditions that make a homogeneous space a geometry in the sense of Thurston. Imitating this approach in the 5-dimensional case, the problem divides into cases for each representation by which the point stabilizer G_p of $p \in M = G/G_p$ acts on the tangent space T_pM (the “linear isotropy representation”).

If T_pM decomposes in a nice way into lower-dimensional irreducible sub-representations, then this decomposition can be exploited to classify the resulting 5-dimensional geometries, via a G -invariant fiber bundle structure on M . Such an approach is carried out in Part III [Gen16]. The present paper concerns itself with the classification when T_pM is irreducible or trivial; in this case, T_pM has no proper characteristic summands as an G_p -representation, which makes its decomposition less useful.

Instead, we appeal to existing classifications: the classification of strongly isotropy irreducible homogeneous spaces by Manturov [Man61a, Man61b, Man66, Man98], Wolf [Wol68, Wol84], and Krämer [Krä75] when T_pM is irreducible; and the method used by Mubarakzyanov in [Mub63] and Filipkiewicz in [Fil83, Ch. 6] for classifying unimodular solvable real Lie algebras by nilradical when T_pM is trivial. Adapting these classifications to the setting of Thurston's geometries requires answering questions about lattices and maximal isometry groups, which is done in this paper. The result is as follows.

Theorem 1.1 (Classification of 5-dimensional maximal model geometries with irreducible or trivial isotropy). *Let $M = G/G_p$ be a 5-dimensional maximal model geometry.*

- (i) *If G_p acts irreducibly on the tangent space T_pM , then M is one of the classical spaces \mathbb{E}^5 , S^5 , and \mathbb{H}^5 with its usual isometry group, or one of the other irreducible Riemannian symmetric spaces $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ and $\mathrm{SU}(3)/\mathrm{SO}(3)$. (Prop. 4.1)*
- (ii) *If G_p acts trivially on T_pM , then $M \cong G$ is one of the following connected, simply-connected, solvable Lie groups.*
 - (a) $\mathbb{R}^3 \rtimes \{xyz = 1\}^0$, the semidirect product $\mathbb{R}^3 \rtimes \mathbb{R}^2$ where \mathbb{R}^2 acts by diagonal matrices with positive entries and determinant 1. (Prop. 5.12)
 - (b) $\mathbb{R}^4 \rtimes_{e^{tA}} \mathbb{R}$, where the multiset of characteristic polynomials of the Jordan blocks of A is one of the multisets below. These are abbreviated as $\mathbb{R}^4 \rtimes_{\text{polynomials}} \mathbb{R}$. (Prop. 5.19)
 - $\{x - a, x - b, x - c, x + a + b + c\}$, where
 1. $a \neq b \neq c \neq a$, and
 2. e^{tA} has integer characteristic polynomial for some t .
(This is a countably infinite family of geometries.)
 - $\{x^2, x - 1, x + 1\}$
 - $\{(x - 1)^2, (x + 1)^2\}$
 - $\{x^3, x\}$
 - $\{x^4\}$

(c) $\text{Nil}^4 \rtimes \mathbb{R}$ and $\text{Nil}^4 \rtimes \mathbb{R}$, whose Lie algebras have basis x_1, \dots, x_5 and the following nonzero brackets. (Prop. 5.22)

$$\begin{aligned} [x_4, x_3] &= x_2 & [x_5, x_3] &= x_1 \\ [x_4, x_2] &= x_1 & [x_5, x_4] &= 0 \text{ or } x_3 \text{ respectively.} \end{aligned}$$

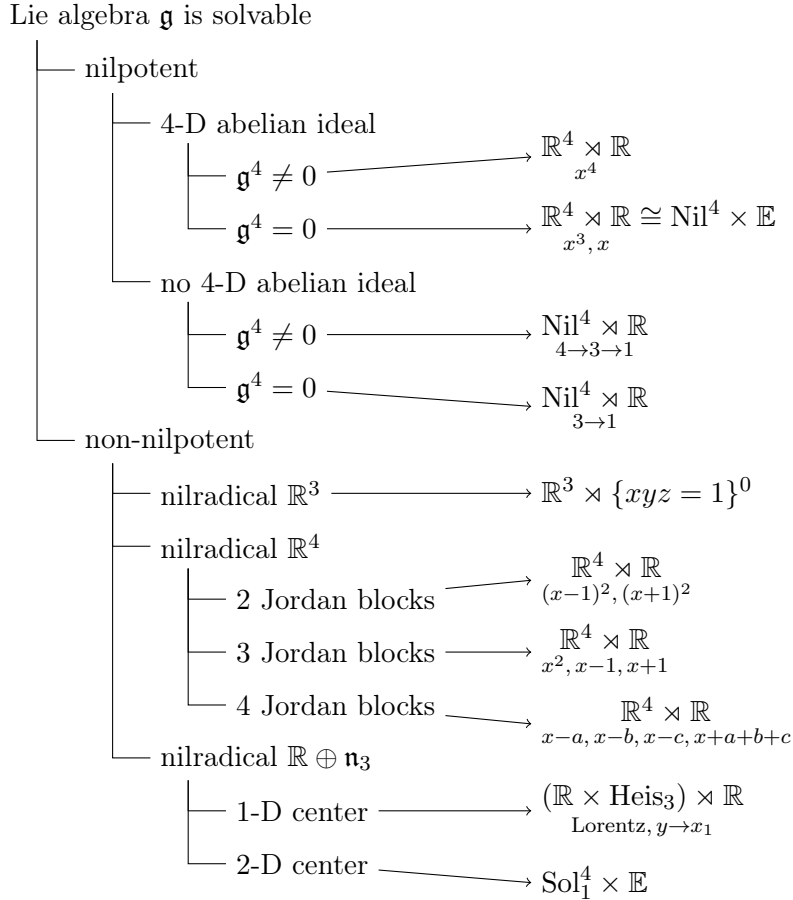
(d) $\mathbb{R} \times \text{Sol}_1^4$ and $(\mathbb{R} \times \text{Heis}_3) \rtimes \mathbb{R}$, whose Lie algebras have basis y, x_1, x_2, x_3, z and the following nonzero brackets. (Prop. 5.25)

$$[x_3, x_2] = x_1 \quad [z, x_2] = x_2 \quad [z, x_3] = -x_3 \quad [z, y] \mapsto 0 \text{ or } x_1 \text{ respectively.}$$

Moreover, all of the above spaces are maximal model geometries.

Remark 1.2. Properties of the geometries are not explored in depth here. For the symmetric spaces in Thm. 1.1(i), one can see e.g. [Wol11, §9.6] for a discussion of $\text{SU}(3)/\text{SO}(3)$ and [BGS85, Appendix 5] for a discussion of $\text{SL}(3, \mathbb{R})/\text{SO}(3)$. To make sense of the array of solvable Lie groups in Thm. 1.1(ii), one can construct an identification key to distinguish them, such as Figure 1.3 (proven in Prop. 5.31).

Figure 1.3: Identification key for geometries $G = G/\{1\}$.



Remark 1.4. When the group is a direct product with \mathbb{R} , the geometry is written as a product with \mathbb{E} , following Thurston's convention in [Thu97, Thm. 3.8.4] to highlight that the \mathbb{R} factor behaves as a 1-dimensional Euclidean space.

Some of the $\mathbb{R}^4 \rtimes \mathbb{R}$ geometries can also be named as products of lower-dimensional geometries, using names from [Wal86, Table 1]. If the polynomials are x^3 and x , then the Lie group is isomorphic to $\text{Nil}^4 \times \mathbb{R}$. If there are 4 polynomials and one of them is x , then the Lie group is isomorphic to $\text{Sol}_{m,n}^4 \times \mathbb{R}$.

An alternative naming scheme is given in [PSWZ76, Table II], from Mubarakzhanov's classification [Mub63] of solvable Lie algebras; Table 1.5 shows the concordance.

Table 1.5: Lie algebra names from [PSWZ76, Tables I, II] for geometries with trivial isotropy.

Geometry	Lie algebra
$\text{Nil}^4 \times \mathbb{E}$	$A_{4,1} \oplus \mathbb{R}$
$\text{Sol}_1^4 \times \mathbb{E}$	$A_{4,8} \oplus \mathbb{R}$
$\mathbb{R}^4 \rtimes \mathbb{R}$ x^4	$A_{5,2}$
$\text{Nil}^4 \rtimes \mathbb{R}$ $3 \rightarrow 1$	$A_{5,5}$
$\text{Nil}^4 \rtimes \mathbb{R}$ $4 \rightarrow 3 \rightarrow 1$	$A_{5,6}$
$\mathbb{R}^4 \rtimes \mathbb{R}$ $x-a, x-b, x-c, x+a+b+c$	$A_{5,7}^{a,b,c} \ (a+b+c = -1)$
$\mathbb{R}^4 \rtimes \mathbb{R}$ $x^2, x-1, x+1$	$A_{5,8}^{-1}$
$\mathbb{R}^4 \rtimes \mathbb{R}$ $(x-1)^2, (x+1)^2$	$A_{5,15}^{-1}$
$(\mathbb{R} \times \text{Heis}_3) \rtimes \mathbb{R}$ Lorentz, $y \rightarrow x_1$	$A_{5,20}^0$
$\mathbb{R}^3 \rtimes \{xyz = 1\}^0$	$A_{5,33}^{-1,-1}$

Section 2 collects some basic facts about geometries and homogeneous spaces. Section 3 classifies the isotropy representations (point stabilizers). Sections 4 and 5 carry out the classification of geometries when the isotropy is irreducible or trivial, respectively.

2 General background: geometries

This section recalls terms and basic facts about geometries, including their interpretations as Riemannian manifolds or homogeneous spaces. This starts with the definition of a geometry, following Thurston in [Thu97, Defn. 3.8.1] and Filipkiewicz in [Fil83, §1.1].

Definition 2.1 (Geometries).

- (i) A *geometry* is a pair $(M, G \subseteq \text{Diff } M)$ where M is a connected, simply-connected smooth manifold and G is a connected Lie group acting transitively, smoothly, and with compact point stabilizers.

- (ii) (M, G) is a *model geometry* if there is a finite-volume complete Riemannian manifold N locally isometric to M with some G -invariant metric. Such an N is said to be *modeled on* (M, G) .
- (iii) (M, G) is *maximal* if there is no geometry (M, H) with $G \subsetneq H$. Any such (M, H) is said to *subsume* (M, G) .

As Thurston remarks in [Thu97, §3.8], M can be thought of as a Riemannian manifold as long as one is willing to change the metric. Since the tools for proving maximality are stated in the language of Riemannian geometry, here is the explicit relationship with Riemannian manifolds.

Proposition 2.2 (Existence of invariant metric; see e.g. [Thu97, Prop. 3.4.11]). *Suppose a Lie group G acts transitively, smoothly, and with compact point stabilizers on a smooth manifold M . Then M has a G -invariant Riemannian metric.*

This makes G a subgroup of the isometry group. The next Proposition asserts this inclusion is an equality for maximal geometries, which makes it possible to approach questions of maximality by leveraging existing results describing isometry groups of various spaces.

Proposition 2.3 (Description of maximality, [Fil83, Prop. 1.1.2]).

- (i) *Every geometry is maximal or subsumed by a maximal geometry.*
- (ii) *If (M, G) is a maximal geometry, then G is the identity component $(\text{Isom } M)^0$ of $\text{Isom } M$ in any G -invariant metric on M .*

Remark 2.4. By connectedness, if (M, G) is realized by (M, H) with $G \subsetneq H$, then $\dim G < \dim H$ and $\dim G_p < \dim H_p$. Hence to verify maximality of a geometry (M, G) , it suffices to distinguish (M, G) from the geometries whose point stabilizers contain G_p . Knowing inclusions between possible point stabilizers (Figure 3.2) will facilitate this.

The interpretation of geometries as homogeneous spaces is already integral to the 4-dimensional classification by Filipkiewicz, and some geometries are most concisely expressed as homogeneous spaces. So the dictionary is provided explicitly below, with an outline of the proof and prerequisites in case the reader requires details.

Proposition 2.5 (Geometries described as homogeneous spaces).

- (i) *Geometries (M, G) correspond one-to-one with simply-connected homogeneous spaces G/G_p where G is a connected Lie group and G_p is compact and contains no nontrivial normal subgroups of G . The correspondence is $G/G_p \leftrightarrow (G/G_p, G)$.*
- (ii) *G/G_p is a model geometry if and only if some lattice $\Gamma \subset G$ intersects no conjugate of G_p nontrivially.*
- (iii) *G/G_p is a maximal geometry if and only if it is not G -equivariantly diffeomorphic to a geometry G'/G'_p with $G \subsetneq G'$.*

Proof. Part (i) follows from two standard facts:

1. If a Lie group G acts transitively on a smooth manifold M with subgroup G_p stabilizing $p \in M$, then G/G_p with its natural smooth structure is G -equivariantly diffeomorphic to M [Fil83, p. 1]. (See also [Hel78, Thm. II.3.2, Prop. II.4.3(a)], [GOV93, Thm. II.1.1.2], or [Wol11, 1.5.9].)

2. A group G acts faithfully on the coset space G/G_p (i.e. is a subgroup of $\text{Diff}(G/G_p)$, as opposed to merely surjecting onto one) if and only if G_p contains no nontrivial normal subgroups of G . (Proof: The kernel of $G \rightarrow \text{Diff}(G/G_p)$ is normal and contained in G_p since it fixes the identity coset. Conversely, if $N \subseteq G_p$ is normal in G , then $G \rightarrow \text{Diff}(G/G_p)$ factors through G/N .)

Part (ii) is the following argument distilled from [Thu97, §3.3–3.4]. Fix any G -invariant Riemannian metric on $M = G/G_p$. Discrete subgroups Γ correspond to orbifolds $N = \Gamma \backslash M$ covered by M . Such N naturally inherits a Riemannian metric (i.e. is a manifold) if and only if Γ acts freely on M , i.e. Γ intersects no point stabilizer nontrivially. Since G acts transitively on M with compact point stabilizers,

1. every manifold quotient $\Gamma \backslash M$ is complete [Thu97, Cor. 3.5.12 and Prop. 3.4.15];
2. every complete Riemannian manifold N locally isometric to M is isometric to a quotient space $\Gamma \backslash M$ [Thu97, Prop. 3.4.5 and 3.4.15]; and
3. $\Gamma \backslash M$ has finite volume if and only if $\Gamma \backslash G$ does (i.e. Γ is a lattice). (Proof: Let ω be a left-invariant volume form on G . Cosets of G_p are compact, so one may integrate along them, recovering the invariant volume form on M up to a scale factor. Then $\text{vol}(\Gamma \backslash M)$ is a multiple of $\text{vol}(\Gamma \backslash G)$.)

Part (iii) is merely the original definition (2.1(iii)) rephrased in terms of the correspondence from part (i). \square

3 Classification of isotropy representations (point stabilizers)

This section presents the first step in the strategy of Thurston and Filipkiewicz, which is to classify the linear isotropy representations—representations of point stabilizers G_p on tangent spaces $T_p M$.

Since an isometry of a connected Riemannian manifold is determined by its value and derivative at a point (see e.g. [BP92, Prop. A.2.1]), any such action is faithful and preserves the Riemannian metric on $T_p M$. Since G_p is connected by a homotopy exact sequence calculation [Fil83, Prop. 1.1.1] and compact, classifying representations $G_p \curvearrowright T_p M$ is equivalent to classifying closed connected subgroups $G_p \subseteq \text{SO}(5)$, up to conjugacy in $\text{GL}(5, \mathbb{R})$. The classification is as follows, summarized in Figure 3.2.¹

Proposition 3.1 (Classification of isotropy representations). *The closed connected subgroups of $\text{SO}(5)$ are, up to conjugacy in $\text{GL}(5, \mathbb{R})$,*

- *one of the following groups, acting by its standard representation over \mathbb{R} on a subspace of \mathbb{R}^5 and trivially on the orthogonal complement;*

$$\{1\} \quad \text{SO}(2) \quad \text{SO}(3) \quad \text{SO}(4) \quad \text{SO}(5) \quad \text{SO}(2) \times \text{SO}(2) \quad \text{SO}(2) \times \text{SO}(3)$$

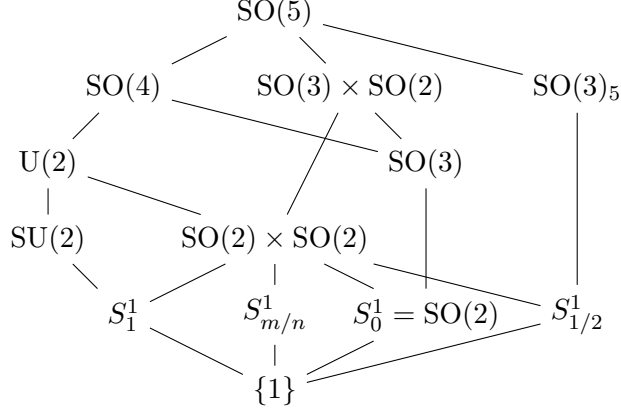
- *one of $\text{SU}(2)$ or $\text{U}(2)$ acting by its standard representation on $\mathbb{C}^2 \cong \mathbb{R}^4 \subset \mathbb{R}^5$;*
- *$\text{SO}(3)_5$, a copy of $\text{SO}(3)$ acting irreducibly on \mathbb{R}^5 ; or*

¹ Proofs of the inclusions are omitted; all are readily guessed except for $S_{1/2}^1 \subset \text{SO}(3)_5$. This last inclusion can be seen from the description of $\text{SO}(3)_5$ as $\text{SO}(3)$ acting by conjugation on traceless symmetric bilinear forms—a 90 degree rotation in the first two coordinates has order 4 but acts with order 2 on the diagonal matrix $d(1, -1, 0)$.

- $S^1_{m/n}$ ($0 \leq \frac{m}{n} \leq 1$), the 1-parameter subgroup of $\text{SO}(2) \times \text{SO}(2)$ defined by

$$S^1 \ni z \mapsto (z^m, z^n) \in S^1 \times S^1 \cong \text{SO}(2) \times \text{SO}(2).$$

Figure 3.2: Closed connected subgroups of $\text{SO}(5)$, with inclusions.



3.1 Proof of the classification of isotropy representations

The proof presented below is similar in spirit to that of [Fil83, §1.2], albeit with somewhat more representation-theoretic language. In higher dimensions, it would be advantageous to use an alternative strategy that provided more structure and fewer opportunities for human error. One such approach, to list maximal subgroups recursively using results by Dynkin, is outlined in [KK13, Tables 1, 3] for several groups including $\text{SO}(5)$ and $\text{SO}(6)$.

In dimension 5, the groups involved are small enough that only representations of S^1 and $\text{SU}(2)$ really need to be understood; this permits a slightly more elementary approach, using some standard facts about representations of low-dimensional groups. These will be stated where used, with proofs deferred to Section 3.2 so that they are available but not a source of clutter.

The first step is the classification up to finite covers by using the classification of compact simple Lie groups.

Proposition 3.3. *Any closed connected proper subgroup $K \subset \text{SO}(5)$ is finitely covered by a product of at most two factors, each of which is S^1 or $\text{SU}(2)$.*

Proof. Every compact connected Lie group is finitely covered by a product of circles S^1 and compact simply-connected simple Lie groups [BD85, Thm. V.8.1]. By the classification of compact connected simple Lie groups [Hel78, Ch. X, §6 (p. 516)], the simple factors could be

- $\text{Sp}(n)$, $n \geq 1$ (dim. $n(2n+1)$)
- $\text{SU}(n)$, $n \geq 3$ (dim. n^2-1)
- $\text{Spin}(n)$, $n \geq 7$ (dim. $\binom{n}{2}$)
- G_2 (dim. 14), F_4 (dim. 52), E_6 (dim 78), E_7 (dim. 133), or E_8 (dim. 248).

The dimension of $\mathrm{SO}(5)$ is 10; so only $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ and $\mathrm{SU}(3)$ could be simple factors of a cover of a proper subgroup of $\mathrm{SO}(5)$. Moreover, $\mathrm{SU}(3)$ is not a factor since it has no nontrivial 5-dimensional representations (Prop. 3.4).

Hence K is finitely covered by a product whose factors are S^1 or $\mathrm{SU}(2)$. Since $\mathrm{SO}(5)$ has rank 2, such a product has at most two factors. \square

At this stage, the full classification (Prop. 3.1) reduces to listing low-dimensional representations of S^1 and $\mathrm{SU}(2)$.

Proof of Prop. 3.1 (Classification of closed connected subgroups $K \subseteq \mathrm{SO}(5)$). By Prop. 3.3 above, a closed connected subgroup $K \subseteq \mathrm{SO}(5)$ either is $\mathrm{SO}(5)$ or has a finite cover $\tilde{K} \cong K_1 \times K_2$, where K_1 and K_2 are S^1 or $\mathrm{SU}(2)$. When \tilde{K} is a product, there are the following cases.

Case 1: $\tilde{K} \cong \{1\}$. Then $K = \{1\}$.

Case 2: $\tilde{K} \cong S^1$. Then K is the image of some homomorphism $f : \mathbb{R} \rightarrow \mathrm{SO}(5)$. The eigenvalues of $f(t) \in \mathrm{SO}(5)$ are 1 and two pairs of complex conjugates of norm 1, with homomorphic dependence on t ; so there is some decomposition of \mathbb{R}^5 as $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ in which

$$f(t) = (e^{xt}, e^{yt}, 1) \in \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SO}(1).$$

Since the image is closed, x and y must be rationally dependent; so $K = S^1_{m/n}$ where either $\frac{x}{y}$ or $\frac{y}{x}$ is equal to $\frac{m}{n}$.

Case 3: $\tilde{K} \cong S^1 \times S^1$. Since the two S^1 factors commute, they share eigenspaces. Using $\mathbb{R}^5 \cong \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ as above, K must be a 2-dimensional subgroup of $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SO}(1) \cong \mathrm{SO}(2) \times \mathrm{SO}(2)$.

Case 4: $\tilde{K} \cong \mathrm{SU}(2)$. The classification of irreducible representations of $\mathrm{SU}(2)$ (Prop. 3.5) implies that K is one of $\mathrm{SU}(2)$ acting on \mathbb{C}^2 , $\mathrm{SO}(3)$ acting on \mathbb{R}^3 , and $\mathrm{SO}(3)$ acting irreducibly on \mathbb{R}^5 .

Case 5: $\tilde{K} \cong \mathrm{SU}(2) \times S^1$. This $\mathrm{SU}(2)$ cannot act irreducibly on \mathbb{R}^5 since that action has trivial centralizer (Cor. 3.8). Hence the $\mathrm{SU}(2)$ factor acts as either $\mathrm{SO}(3)$ on \mathbb{R}^3 or $\mathrm{SU}(2)$ on \mathbb{C}^2 .

- If $\mathrm{SU}(2)$ acts as $\mathrm{SO}(3)$ on some $\mathbb{R}^3 \subset \mathbb{R}^5$, then the S^1 factor preserves the decomposition $\mathbb{R}^5 \cong_{\mathrm{SO}(3)} \mathbb{R}^3 \oplus 2\mathbb{R}$ since it commutes with $\mathrm{SO}(3)$. So $K \subseteq \mathrm{S}(\mathrm{O}(3) \times \mathrm{O}(2))$, and by connectivity $K \subseteq \mathrm{SO}(3) \times \mathrm{SO}(2)$. Equality holds since the dimensions match.
- If $\mathrm{SU}(2)$ acts as $\mathrm{SU}(2)$ on some $\mathbb{C}^2 \subset \mathbb{R}^5$, then as above $K \subseteq \mathrm{SO}(4) \times \mathrm{SO}(1) \cong \mathrm{SO}(4)$. Since $\mathrm{SO}(4)$ is covered by $\mathrm{SU}(2) \times \mathrm{SU}(2)$, the S^1 factor can be taken to lie in the second $\mathrm{SU}(2)$ as a 1-parameter subgroup—all of which are conjugate. Thus there is only one action of $\mathrm{SU}(2) \times S^1$ on \mathbb{R}^4 up to conjugacy. It can be realized as the action by $\mathrm{SU}(2)$ on \mathbb{C}^2 along with the action of the unit-norm scalars, which amounts to $\mathrm{U}(2)$.

Case 6: $\tilde{K} \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. As in Case 5 above, $K \subseteq \mathrm{SO}(4)$. Since the dimensions match, $K = \mathrm{SO}(4)$. \square

3.2 Standard facts about representations of lower-dimensional groups

This subsection collects some basic facts about representations of $SU(2)$, $SU(3)$, and $SO(3)$. These are only needed for the above classification of isotropy representations (Prop. 3.1), so a reader familiar with these groups may wish to skip this subsection.

Proposition 3.4. *Representations of $SU(3)$ of dimension less than 6 over \mathbb{R} are trivial.*

Proof. The complex irreducible representations of $\mathfrak{su}_3\mathbb{C} \cong \mathfrak{sl}_3\mathbb{C}$ ² exponentiate to those of the simply-connected group $SU(3)$. Listing highest weights as done in [FH91, Ch. 12] shows that they have dimensions 1, 3, 6, and higher. Over \mathbb{R} , an irreducible representation over \mathbb{C} either remains irreducible or splits into two isomorphic irreducible summands [BD85, Prop. II.6.6(vii-ix)]; so any real irreducible representation has dimension 1, 2, 3, 6, or higher.

Then choosing an invariant inner product, an irreducible action of $SU(3)$ on \mathbb{R}^k ($k < 6$) factors through $SO(3)$. Since $SU(3)$ is simple and of higher dimension than $SO(3)$, its image in $SO(3)$ is trivial. \square

Proposition 3.5. *Let V be the standard representation of $SU(2)$ over \mathbb{C} . The finite-dimensional irreducible representations of $SU(2)$ over \mathbb{R} are*

- (i) *for even $n \geq 0$, an invariant real subspace of $\text{Sym}^n V$ (dimension $n + 1$); and*
- (ii) *for odd $n \geq 1$, the representation $\text{Sym}^n V$ taken as a real vector space (dimension $2(n + 1)$).*

Proof. The irreducible representations of $SU(2)$ over \mathbb{C} are the symmetric powers of V [BD85, Prop. II.5.1, II.5.3] (see also [FH91, 11.8]). The action can be written explicitly as

$$\begin{pmatrix} a & b \\ -\bar{b} & -\bar{a} \end{pmatrix} f(x, y) = f(ax + by, -\bar{b}x + \bar{a}y)$$

where $f \in \text{Sym}^n V$ is homogeneous of degree n in $\mathbb{C}[x, y]$.

Define $\phi : \text{Sym}^n V \rightarrow \text{Sym}^n V$ by

$$(\phi f)(x, y) = \overline{f(\bar{y}, -\bar{x})}.$$

One can verify from this formula that ϕ is $SU(2)$ -equivariant and satisfies $\phi i = -i\phi$ and $\phi^2 = (-1)^n$.

- (i) If n is even, then ϕ is a real structure on $\text{Sym}^n V$. An irreducible representation over \mathbb{C} with a real structure is the direct sum of two copies of an irreducible representation over \mathbb{R} [BD85, Prop. II.6.6(vii)]. A summand can be recovered as $(\phi + \text{Id})(\text{Sym}^n V)$, consisting of the polynomials $f(x, y) = \sum_m f_m x^m y^{n-m}$ where $f_m = (-1)^m \overline{f_{n-m}}$.
- (ii) If n is odd, then ϕ is a quaternionic structure on $\text{Sym}^n V$. An irreducible representation over \mathbb{C} with a quaternionic structure is irreducible over \mathbb{R} [BD85, Prop. II.6.6(ix)]. \square

Remark 3.6 ([BD85, II.5.4]). If and only if n is even, the scalar $-1 \in SU(2)$ acts trivially on $\text{Sym}^n V$, allowing the action of $SU(2)$ to descend to an action of $SO(3) \cong SU(2)/\{\pm 1\} = SU(2)/Z(SU(2))$.

Proposition 3.7. *End $W \cong \mathbb{R}$ for any irreducible representation W of $SO(3)$ over \mathbb{R} .*

Proof. Every endomorphism of W extends to an endomorphism of $W \otimes \mathbb{C}$, which is irreducible over \mathbb{C} . By Schur's lemma, $\phi \otimes \mathbb{C}$ is multiplication by a scalar. Since $\dim W$ is odd (Rmk. 3.6), the only scalars that preserve W are the reals. \square

Corollary 3.8. *The centralizer of $SO(3)_5$ in $SO(5)$ is trivial.*

² For the isomorphism, write $\mathfrak{sl}_n\mathbb{R}$ as the sum of its skew-symmetric part \mathfrak{k} and symmetric part \mathfrak{p} . Then since \mathfrak{su}_n consists of the matrices A which satisfy $A + A^* = 0$ and are traceless, $\mathfrak{su}_n = \mathfrak{k} + i\mathfrak{p}$.

4 The case of irreducible isotropy

When $G_p \curvearrowright T_p M$ is irreducible, the classification of isotropy irreducible homogeneous spaces and the classification of irreducible Riemannian symmetric spaces produce a list (Prop. 4.1) of homogeneous spaces as candidates. Then it only remains to check that each is a maximal model geometry with irreducible isotropy (Prop. 4.4–4.5). Taken together, these results prove Thm. 1.1(i).

4.1 The list of candidates

The first step of this classification is to obtain, from the classification of isotropy irreducible homogeneous spaces, the following explicit list of candidate geometries.

Proposition 4.1. *Let $M = G/G_p$ be a 5-dimensional maximal geometry for which $G_p \curvearrowright T_p M$ is irreducible. Then M is one of the following Riemannian symmetric spaces.*

$$\begin{array}{ll} \mathbb{E}^5 = \mathbb{R}^5 \rtimes \mathrm{SO}(5)/\mathrm{SO}(5) & \\ S^5 = \mathrm{SO}(6)/\mathrm{SO}(5) & \mathrm{SU}(3)/\mathrm{SO}(3) \\ \mathbb{H}^5 = \mathrm{SO}(5,1)/\mathrm{SO}(5) & \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3) \end{array}$$

Aside from consulting existing classifications, only the following standard fact is needed to produce the above list.

Lemma 4.2. *All copies of $\mathrm{SO}(3)$ in $\mathrm{SU}(3)$ are conjugate.*

Proof. The irreducible representations of $\mathrm{SO}(3)$ over \mathbb{C} of dimension 3 or lower are the trivial and the standard representation; so any two embeddings $\mathrm{SO}(3) \hookrightarrow \mathrm{SU}(3)$ are conjugate by some representation isomorphism $h \in \mathrm{GL}(3, \mathbb{C})$.

Isomorphic unitary representations are unitarily isomorphic (see e.g. [BD85, Exercise II.1.8], or a proof in [BR86, Prop. 5.2.1] using polar decomposition). Then h can be taken to lie in $\mathrm{SU}(3)$, so the two embeddings of $\mathrm{SO}(3)$ are conjugate in $\mathrm{SU}(3)$. \square

Proof of Prop. 4.1. Assume $M = G/G_p$ is a 5-dimensional maximal geometry such that $G_p \curvearrowright T_p M$ is irreducible. It is a theorem of Wolf that if a homogeneous space G/G_p has compact, connected, irreducibly-acting isotropy G_p , then either G/G_p is a Riemannian symmetric space or G is a compact simple Lie group [Wol68, Thm. 1.1].³ From the classification of isotropy representations $G_p \curvearrowright T_p M$ (Prop. 3.1), G_p is either $\mathrm{SO}(5)$ or $\mathrm{SO}(3)_5$.

Case 1: $G_p = \mathrm{SO}(5)$. Since $\mathrm{SO}(5)$ acts transitively on 2-planes through the origin in \mathbb{R}^5 , M has constant sectional curvature—and is therefore exactly one of \mathbb{E}^5 , S^5 , and \mathbb{H}^5 by the Killing-Hopf theorem [Wol11, Cor. 2.4.10].

³ In [Wol68, Table, p. 107–110], Wolf gave a more explicit classification of strongly isotropy irreducible spaces. Wang and Ziller remark in [WZ91, p. 2] that this classification has an omission but do not say whether the erratum in [Wol84] completes it. Instead they refer the reader to two other classifications, by Manturov in [Man61a, Man61b, Man66] (earlier, also with omissions; see also [Man98]) and by Krämer in [Krä75] (believed complete). A slightly weaker version of the result used here—omitting the claim that G is simple—was known to Matsushima, with proof first given by Nagano in [Nag59, Appendix].

Case 2: $G_p = \mathrm{SO}(3)_5$. In this case, G is 8-dimensional. If M is an irreducible symmetric space, then M is Euclidean, $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$, or $\mathrm{SU}(3)/\mathrm{SO}(3)$ by the classification of irreducible symmetric spaces (see [Hel78, X.6 Table V and p.515–518]).

Otherwise, G is an 8-dimensional compact simple Lie group. By the classification of compact simple Lie groups [Hel78, X.6 Table IV], $G \cong \mathrm{SU}(3)$. Since all copies of $\mathrm{SO}(3)$ in $\mathrm{SU}(3)$ are conjugate (Lemma 4.2), X is the symmetric space $\mathrm{SU}(3)/\mathrm{SO}(3)$. \square

Remark 4.3. We have not explicitly verified that the candidate spaces have irreducible isotropy. That this holds for every irreducible Riemannian symmetric space (which seems to be well known; see e.g. [Wol68, Ch. 1, condition (v)]) is a side effect of proving the usual decomposition theorem for Riemannian symmetric spaces by using orthogonal involutive Lie algebras. Such an approach can be found in [Wol11, Thm. 8.3.8].

Alternatively, it follows in the case of $\mathrm{SO}(5)$ from $\mathrm{SO}(5)$ having a transitive action on S^4 ; and in the case of $\mathrm{SO}(3)_5$ from observing that $\mathrm{SO}(3)$ acts irreducibly on the space V of traceless symmetric 3×3 matrices, and writing

$$\mathfrak{sl}_3\mathbb{R} = \mathfrak{so}_3\mathbb{R} + V \qquad \mathfrak{su}_3\mathbb{R} = \mathfrak{so}_3\mathbb{R} + iV.$$

4.2 Maximality and the existence of compact quotients

It happens that all non-Euclidean isotropy irreducible spaces—with two exceptions, neither of which has dimension 5—are already known to be maximal model geometries. So to prove that the 5-dimensional isotropy irreducible geometries (i.e. those produced in Prop. 4.1) are maximal model geometries, it suffices to collect some existing theorems.

Proposition 4.4. *Any geometry with irreducible isotropy is a model geometry (i.e. admits a compact manifold quotient).*

Proof. As part of the classification of strongly isotropy irreducible spaces, such a geometry is either already compact or Riemannian symmetric [Wol68, Thm. 1.1]. Borel proved in [Bor63, Thm. A] that every simply-connected Riemannian symmetric space G/K admits a compact manifold quotient $\Gamma \backslash G/K$. \square

Since all of the 5-dimensional isotropy irreducible geometries are Riemannian symmetric spaces, it suffices to know when a Riemannian symmetric space is maximal. The geometry \mathbb{E}^5 is maximal since its isotropy $\mathrm{SO}(5)$ is maximal (Rmk. 2.4); and the other candidates G/K have semisimple G , so the following result proves them maximal.

Proposition 4.5. *Suppose G/K is a Riemannian symmetric space—i.e. suppose K is an open subgroup of the fixed set in G of some order 2 element of $\mathrm{Aut} G$. Suppose further that G is semisimple and acts faithfully on G/K . Then G/K is a maximal geometry.*

Proof. If G/K is a Riemannian symmetric space with G semisimple and acting faithfully, then $G = (\mathrm{Isom} G/K)^0$ in every G -invariant metric on G/K [Hel78, Thm. V.4.1(i)]. At least one of these invariant metrics has an isometry group whose identity component is the transformation group G' of a maximal geometry G'/K' realizing G/K (Prop. 2.3). Then $G' = G$, so G/K is maximal. \square

Remark 4.6. Wolf also proved maximality in the non-symmetric case in [Wol68, Thm. 17.1]. That is, except for $G_2/\mathrm{SU}(3) \cong S^6$ and $\mathrm{Spin}(7)/G_2 \cong S^7$, a simply-connected isotropy irreducible Riemannian homogeneous space G/K is maximal if G is semisimple (i.e. G/K is not Euclidean) and acts faithfully. The same theorem also includes a description of the full isometry group—not just the identity component.

Remark 4.7. One could instead verify maximality by checking that the listed spaces are not extended by any geometries with larger isotropy groups. The constant-curvature geometries have maximal isotropy; $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ is distinguished from the constant-curvature geometries by having rank 2 [BGS85, Appendix 5 §2, p. 242]; and $\mathrm{SU}(3)/\mathrm{SO}(3)$ is distinguished by having nonzero π_2 , which can be calculated using the homotopy exact sequence.

5 The case of trivial isotropy: solvable Lie groups

This section proves Thm. 1.1(ii), the classification of 5-dimensional maximal model geometries $M = G/G_p$ for which $G_p \curvearrowright T_p M$ is trivial. (The reader may wish to consult the identification key in Figure 1.3 for a reminder of the results.)

Overview (see also “Roadmap” below). Our strategy, following that of Filipkiewicz in [Fil83, §6], begins by invoking Filipkiewicz’s reduction to a classification of simply-connected solvable groups.

Proposition 5.1 ([Fil83, Prop. 6.1.3]). *If $M = G/G_p$ is a maximal model geometry with 0-dimensional point stabilizers, then $M \cong G$ is a connected, simply-connected, unimodular solvable Lie group, and $\mathrm{Aut}(G)$ is solvable and simply-connected.*

The classification proceeds by expressing G as an extension of an abelian group by a nilpotent group (such as the nilradical). Conveniently, only split extensions are needed in order to produce the maximal geometries. That is, Section 5.2 will prove that

Proposition 5.2. *If $G = G/\{1\}$ is a maximal model geometry of dimension 5, then either*

- (i) $G \cong \mathbb{R}^3 \rtimes \mathbb{R}^2$ where \mathbb{R}^2 acts on \mathbb{R}^3 as the diagonal matrices with positive entries and determinant 1; or
- (ii) $G \cong N \rtimes \mathbb{R}$ where N is nilpotent, connected, and simply-connected.

The problem then reduces to classifying semidirect products and checking lattice existence (for model geometries) and maximality. We perform this classification in the language of Lie algebras, using the correspondence between Lie algebras and connected, simply-connected Lie groups (see e.g. [GOV93, Thm. I.2.2.10–11] and [GOV94, Thm. 1.4.2]).

Roadmap. After Section 5.1 lists some notation, Section 5.2 proves the above proposition using some Lie algebra cohomology. Details on the $\mathbb{R}^3 \rtimes \mathbb{R}^2$ geometry (in (i)) are in Section 5.3, including an application of Dirichlet’s unit theorem (Prop. 5.16). For each of the three groups that can occur as N in (ii), a subsection of Section 5.4 lists semidirect products, omitting any that are easily shown not to produce a maximal model geometry; and questions of lattice existence determine the model geometries. Section 5.5 proves maximality (Prop. 5.30) using a general theorem by Gordon and Wilson, and lists features distinguishing the geometries from each other (Prop. 5.31). Taken together, these results prove Thm. 1.1(ii).

Remark 5.3. The 5-dimensional solvable Lie algebras over \mathbb{R} having already been classified by Mubarakzhanov in [Mub63] using a largely similar approach,⁴ much of this case could be reduced to consulting a table such as [PSWZ76, Table II]. It is instead presented explicitly here, since the

⁴The classification is complete in dimensions ≤ 6 and is known in limited cases for higher dimensions; see [ŠW12, Introduction] for a survey.

method stands on its own and is illustrative—exposing tools and calculations that will be reused in classifying the fibered geometries (particularly a partial classification of extensions of \mathbb{R}^2 by \mathbb{R}^3 in Lemmas 5.13 and 5.15). Only a fraction of the classification of solvable Lie algebras over \mathbb{R} is duplicated, since only those which admit lattices and are tangent to maximal geometries are of interest.

5.1 Notations

The strategy outlined above for classifying solvable Lie groups involves nilpotent subalgebras of their Lie algebras. The following two definitions will aid in naming such subalgebras.

Definition 5.4. \mathfrak{n}_k is the semidirect sum $\mathbb{R}^{k-1} \rtimes \mathbb{R}$ where some $x_k \in \mathbb{R}$ acts on \mathbb{R}^{k-1} in its standard basis $\{x_1, \dots, x_{k-1}\}$ by a single Jordan block with eigenvalue 0.

Example 5.5. \mathfrak{n}_3 is the 3-dimensional Heisenberg Lie algebra, and \mathfrak{n}_4 is the unique 4-dimensional indecomposable nilpotent Lie algebra (see [PSWZ76, Table I] or Prop. 5.18 below).

Definition 5.6 (Nilradical, see e.g. [GOV94, §2.5]). The *nilradical* $\text{nil}(\mathfrak{g})$ of a Lie algebra \mathfrak{g} is the unique maximal nilpotent ideal of \mathfrak{g} .

Lie algebra extensions that split are semidirect sums $\mathfrak{h} \rtimes \mathfrak{g}$, which are classified by the action $\mathfrak{g} \rightarrow \text{der}(\mathfrak{h})/\text{ad}(\mathfrak{h}) = \text{out}(\mathfrak{h})$. As the actions encountered are usually traceless due to unimodularity considerations (such as in Lemma 5.9 below), we also make the following definition.

Definition 5.7. If \mathfrak{h} is a unimodular Lie algebra (so that $\text{ad } \mathfrak{h}$ acts tracelessly), $\text{sout } \mathfrak{h} \subseteq \text{out } \mathfrak{h}$ denotes the subalgebra consisting of traceless outer derivations.

5.2 Reduction to semidirect products

This section proves Prop. 5.2, which asserts that a maximal model geometry $G = G/\{1\}$ of dimension 5 is one of two forms of semidirect product. Since such a claim bears a close resemblance to [Fil83, Prop. 6.1.4] from the 4-dimensional case, an n -dimensional generalization such as the following may be of interest.

Proposition 5.8 (G is an extension of an abelian group by a nilpotent group). *Suppose \mathfrak{g} is a unimodular solvable Lie algebra of dimension $n > 1$.*

(i) *If \mathfrak{g} is not nilpotent then \mathfrak{g} is an extension*

$$0 \rightarrow \text{nil } \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow \mathbb{R}^k \rightarrow 0$$

for some $0 < k < n$, and \mathfrak{g} acts tracelessly on $\text{nil } \mathfrak{g}$.

(ii) *If \mathfrak{g} is nilpotent, then it is a semidirect sum of a nilpotent ideal \mathfrak{n} and \mathbb{R} , with \mathbb{R} acting tracelessly.*

The following outline summarizes the gap between the above extension problems and the two semidirect products described in Prop. 5.2.

Proof outline of Prop. 5.2. The Lie algebra \mathfrak{g} of a model geometry $G = G/\{1\}$ is unimodular and solvable (Prop. 5.1). The nilradical $\text{nil } \mathfrak{g}$ of a 5-dimensional unimodular solvable Lie algebra \mathfrak{g} is either \mathbb{R}^3 or of dimension at least 4 (Prop. 5.10). The two cases of Prop. 5.2 are proven as follows.

- (i) If $\text{nil } \mathfrak{g} \cong \mathbb{R}^3$ and G is a maximal geometry, then the extension in Prop. 5.8(i) above is split and \mathbb{R}^2 acts on \mathbb{R}^3 by traceless diagonal matrices (Prop. 5.12).
- (ii) If $\text{nil } \mathfrak{g}$ has dimension at least 4, then \mathfrak{g} is an extension of \mathbb{R} by a nilpotent algebra by Prop. 5.8 above; and the extension splits since any linear map from \mathbb{R} is a homomorphism.

The requirement that geometries are simply-connected then allows these Lie algebra results to apply to the corresponding Lie groups. \square

To complete the proof, the following subsections each prove one component—the n -dimensional extension problem (Prop. 5.8), the restriction on $\text{nil } \mathfrak{g}$ (Prop. 5.10), and the case when $\text{nil } \mathfrak{g} = \mathbb{R}^3$ (Prop. 5.12).

5.2.1 The general extension problem

This section proves Prop. 5.8—the description of unimodular solvable Lie algebras as extensions of abelian algebras by nilpotent algebras. The claims about actions being traceless will be proven using the following observation.

Lemma 5.9. *If \mathfrak{a} is an ideal in a unimodular solvable Lie algebra \mathfrak{g} such that $\mathfrak{g}/\mathfrak{a}$ is unimodular, then \mathfrak{g} acts tracelessly on \mathfrak{a} .*

Proof. Suppose $g \in \mathfrak{g}$. Since \mathfrak{g} is unimodular, $\text{tr ad}_{\mathfrak{g}} g = 0$. Since $\mathfrak{g}/\mathfrak{a}$ is unimodular, $\text{tr ad}_{\mathfrak{g}/\mathfrak{a}} g = 0$. The conclusion follows from

$$\text{tr ad}_{\mathfrak{g}} = \text{tr ad}_{\mathfrak{g}/\mathfrak{a}} + \text{tr ad}_{\mathfrak{a}}. \quad \square$$

Armed with this, let \mathfrak{g} be a unimodular solvable Lie algebra that we hope to write as an extension; the proof of Prop. 5.8 divides into the following two almost-independent cases.

Proof of Prop. 5.8(i) (the non-nilpotent case). The derived algebra of a finite-dimensional solvable Lie algebra is contained in the nilradical (see e.g. Chevalley’s theorem, [Jac62, II.7 Thm. 13]),⁵ so $\mathfrak{g}/\text{nil}(\mathfrak{g})$ is abelian, and thus unimodular. Then $\mathfrak{g}/\text{nil}(\mathfrak{g})$ is some \mathbb{R}^k acting tracelessly on $\text{nil } \mathfrak{g}$ (Lemma 5.9). Since $k = 0$ or $k = n$ would make \mathfrak{g} nilpotent, $0 < k < n$. \square

Proof of Prop. 5.8(ii) (the nilpotent case). Since \mathfrak{g} is nilpotent, $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Then any proper vector subspace \mathfrak{n} of \mathfrak{g} containing $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent ideal. Taking \mathfrak{n} to be of codimension 1 makes $\mathfrak{g}/\mathfrak{n} \cong \mathbb{R}$, so \mathfrak{g} is an extension

$$0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{g} \rightarrow \mathbb{R} \rightarrow 0.$$

As an extension of \mathbb{R} , this splits (any section of $\mathfrak{g} \rightarrow \mathbb{R}$ as a linear map is immediately a homomorphism); so $\mathfrak{g} \cong \mathfrak{n} \rtimes \mathbb{R}$. This action is traceless by Lemma 5.9. \square

5.2.2 Nilradicals

This section proves the following restriction on nilradicals of \mathfrak{g} .

Proposition 5.10. *The nilradical of a 5-dimensional unimodular solvable Lie algebra is either \mathbb{R}^3 or of dimension at least 4.*

Nilradicals of dimension 4 are classified later, in Prop. 5.18. The upcoming proof makes use of one technical lemma—that the action of $\mathfrak{g}/\text{nil}(\mathfrak{g})$ on $\text{nil}(\mathfrak{g})$ is somehow morally as good as faithful.

⁵ Alternatively, one could use Lie’s theorem that \mathfrak{g} has a faithful representation as upper-triangular matrices.

Lemma 5.11. *Suppose \mathfrak{g} is a finite-dimensional solvable Lie algebra with $\mathfrak{g}/\text{nil}(\mathfrak{g}) \cong \mathbb{R}^k$. If $f : \mathbb{R}^k \rightarrow \mathfrak{g}$ is any section of the quotient map as a map of vector spaces, then*

$$\text{ad}|_{\text{nil}(\mathfrak{g})} \circ f : \mathbb{R}^k \rightarrow \text{der}(\text{nil}(\mathfrak{g}))$$

is injective.

Proof. Let \mathfrak{k} be the kernel of this map, and let \mathfrak{h} be the vector subspace $\text{nil}(\mathfrak{g}) + f(\mathfrak{k})$ of \mathfrak{g} . Since $\mathfrak{g}/\text{nil}(\mathfrak{g})$ is abelian,

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \text{nil} \mathfrak{g} \subseteq \mathfrak{h};$$

so \mathfrak{h} is an ideal. This inclusion also implies $[\mathfrak{h}, \mathfrak{h}] \subseteq \text{nil} \mathfrak{g}$, which is the base case for the following induction that shows \mathfrak{h} is nilpotent.

$$\mathfrak{h}^{i+1} = [\mathfrak{h}, \mathfrak{h}^i] \subseteq [\text{nil}(\mathfrak{g}) + f(\mathfrak{k}), \text{nil}(\mathfrak{g})^{i-1}] \subseteq \text{nil}(\mathfrak{g})^i + 0$$

Then $\mathfrak{h} = \text{nil} \mathfrak{g}$ by the definition of the nilradical, so $\mathfrak{k} = 0$. □

Lemma 5.11 will also be used in a later section for the classification in the case $\text{nil} \mathfrak{g} = \mathbb{R}^3$. For now, its role is to provide constraints on dimension in the proof of Prop. 5.10.

Proof of Prop. 5.10. The two nilpotent Lie algebras of dimension 3 are \mathbb{R}^3 and \mathfrak{n}_3 (see e.g. [FH91, Lec. 10], [PSWZ76, Table I], or [Mac99, Table 21.3]); so it will suffice to show that $\text{nil} \mathfrak{g}$ has dimension at least 3 and is not \mathfrak{n}_3 .

Step 1: $\text{nil} \mathfrak{g}$ has dimension at least 3. This follows from a bound by Mubarakzhanov on the dimension of the nilradical (see e.g. [GOV94, Thm. 2.5.2]) but can also be proven directly, as follows.

First, $\text{nil}(\mathfrak{g}) \neq 0$ since that would imply $\mathfrak{g} = \mathfrak{g}/\text{nil}(\mathfrak{g}) = \mathbb{R}^5$, which has nilradical \mathbb{R}^5 . Also, $\text{nil} \mathfrak{g} \not\cong \mathbb{R}$ since Lemma 5.11 would then require some injective linear map

$$\mathfrak{g}/\text{nil}(\mathfrak{g}) \cong \mathbb{R}^4 \rightarrow \text{der} \mathbb{R} \cong \mathfrak{gl}_1 \mathbb{R} \cong \mathbb{R}.$$

Similarly, if $\dim \text{nil} \mathfrak{g} = 2$, then any linear section of $\mathfrak{g} \rightarrow \mathfrak{g}/\text{nil}(\mathfrak{g}) \cong \mathbb{R}^3$ would induce an injective linear map $\mathbb{R}^3 \rightarrow \mathfrak{gl}_2 \mathbb{R}$. In fact this map would have to land in $\mathfrak{sl}_2 \mathbb{R}$ since unimodularity of \mathfrak{g} and $\mathfrak{g}/\text{nil}(\mathfrak{g}) \cong \mathbb{R}^k$ requires \mathfrak{g} to act tracelessly (Lemma 5.9). Then $\mathfrak{sl}_2 \mathbb{R}$ would occur as a subalgebra of \mathfrak{g} —in which case \mathfrak{g} would not be solvable since every term of its derived series would contain $\mathfrak{sl}_2 \mathbb{R}$. Hence $\dim \text{nil} \mathfrak{g} \neq 2$.

Step 2: The traceless outer derivation algebra $\text{sout}(\mathfrak{n}_3)$ is the isomorphic image of some $\mathfrak{sl}_2 \mathbb{R} \subset \text{der}(\mathfrak{n}_3)$. Let \mathfrak{n}_3 have basis x_1, x_2, x_3 where x_1 is central and $[x_3, x_2] = x_1$.

A derivation $D : \mathfrak{n}_3 \rightarrow \mathfrak{n}_3$ induces a linear map $\mathbb{R}x_2 + \mathbb{R}x_3 \rightarrow \mathfrak{n}_3/(\mathbb{R}x_2 + \mathbb{R}x_3) \cong \mathbb{R}x_1$. These are in bijection with the inner derivations, so up to subtracting an inner derivation $D(\mathbb{R}x_2 + \mathbb{R}x_3) \subset \mathbb{R}x_2 + \mathbb{R}x_3$. Then

$$Dx_1 = D[x_3, x_2] = [x_3, Dx_2] + [Dx_3, x_2],$$

so relative to the basis $\{x_1, x_2, x_3\}$, the matrix of D is

$$\begin{pmatrix} a+d & & \\ & a & b \\ & c & d \end{pmatrix}.$$

If D is traceless, then $a+d=0$. Then $\text{sout}(\mathfrak{n}_3) \cong \mathfrak{sl}_2 \mathbb{R}$, and a section of $\text{sder}(\mathfrak{n}_3) \rightarrow \text{sout}(\mathfrak{n}_3)$ is given by the above matrix.

Step 3: \mathfrak{n}_3 is not $\text{nil}(\mathfrak{g})$. An extension

$$0 \rightarrow \mathfrak{n}_3 \rightarrow \mathfrak{g} \rightarrow \mathbb{R}^2 \rightarrow 0$$

defines a map $\mathbb{R}^2 \rightarrow \text{sout}(\mathfrak{n}_3)$ by lifting to \mathfrak{g} and taking brackets. Composition with the section from Step 2 produces a homomorphism $\phi : \mathbb{R}^2 \rightarrow \mathfrak{sl}_2\mathbb{R}$. Since $\mathfrak{sl}_2\mathbb{R}$ admits no 2-dimensional abelian subalgebras (any such would make $[\cdot, \cdot] : \Lambda^2 \mathfrak{sl}_2\mathbb{R} \rightarrow \mathfrak{sl}_2\mathbb{R}$ fail to be surjective), ϕ has nonzero kernel. Then \mathfrak{n}_3 cannot be the nilradical of \mathfrak{g} —since if it were, Lemma 5.11 would require ϕ to be injective. \square

5.2.3 The elimination of non-split extensions

The last main ingredient in Prop. 5.2 is the elimination of non-split extensions of \mathbb{R}^2 by \mathbb{R}^3 and extensions with actions other than the one specified, as follows.

Proposition 5.12. *Suppose $G = G/\{1\}$ is a maximal model geometry of dimension 5 and \mathfrak{g} is its Lie algebra. If $\text{nil } \mathfrak{g} \cong \mathbb{R}^3$, then \mathfrak{g} is the semidirect sum $\mathbb{R}^3 \rtimes \mathbb{R}^2$ where \mathbb{R}^2 acts by traceless diagonal matrices.*

The proof makes use of two main computations: the classification of faithful actions $\mathbb{R}^2 \hookrightarrow \mathfrak{sl}_3\mathbb{R}$ (Lemma 5.13) and the classification of extensions using Lie algebra cohomology (Lemma 5.15). These are carried out below, followed by the proof of Prop. 5.12.

Lemma 5.13. *Up to linear changes of coordinates in \mathbb{R}^3 and \mathbb{R}^2 , there are six embeddings $\phi : \mathbb{R}^2 \rightarrow \mathfrak{sl}_3\mathbb{R}$; each sends $(x, y) \in \mathbb{R}^2$ to one of the following matrices. Blank entries are zero.*

$$\begin{pmatrix} 0 & x & y \\ & 0 & x \\ & & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & & y \\ & 0 & x \\ & & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & x & y \\ & 0 & \\ & & 0 \end{pmatrix} \quad \begin{pmatrix} x & y & \\ & x & \\ & & -2x \end{pmatrix} \quad \begin{pmatrix} x & y & \\ -y & x & \\ & & -2x \end{pmatrix} \quad \begin{pmatrix} x & & \\ & y & \\ & & -x-y \end{pmatrix}$$

Proof. Name the above embeddings ϕ_1 through ϕ_6 , and let $\{e_1, e_2\}$ be a basis for \mathbb{R}^2 . Suppose $\phi : \mathbb{R}^2 \rightarrow \mathfrak{sl}_3\mathbb{R}$ is an embedding. The strategy is to find the Jordan form for $\phi(e_1)$ and determine what matrices commute with it in $\mathfrak{sl}_3\mathbb{R}$.

Case 1: $\phi(\mathbb{R}^2)$ contains no matrices with nonzero eigenvalues. Changing coordinates to put $\phi(e_1)$ in Jordan form (or something like it), and then computing centralizers, either

$$\left\{ \begin{array}{l} \phi(e_1) = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \\ \phi(e_2) = \begin{pmatrix} a & b & c \\ & a & b \\ & & a \end{pmatrix} \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} \phi(e_1) = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix} \\ \phi(e_2) = \begin{pmatrix} a & b & c \\ & e & d \\ & & a \end{pmatrix} \end{array} \right\}.$$

Since the image of ϕ consists of traceless matrices with no nonzero eigenvalues, either

$$\phi = \phi_1 \quad \text{or} \quad \phi(e_2) = \begin{pmatrix} 0 & b & c \\ & 0 & d \\ & & 0 \end{pmatrix}.$$

In the latter case, by replacing e_2 with $e_2 - ce_1$ we may assume $c = 0$. Then if both b and d are nonzero, we may rescale coordinates in \mathbb{R}^3 to obtain $\phi_1(e_1)$ (so ϕ is conjugate to ϕ_1). Otherwise, ϕ is conjugate to ϕ_2 or ϕ_3 .

The embeddings ϕ_1 , ϕ_2 , and ϕ_3 are distinct, distinguished by the ranks and nullities of the matrices in their images.

Case 2: $\phi(\mathbb{R}^2)$ contains a matrix with a nonzero eigenvalue. Change coordinates in \mathbb{R}^2 to assume $\phi(e_1)$ has 1 as an eigenvalue. Then it must have one of the following Jordan forms, for some $y \in \mathbb{R}$.

$$\begin{pmatrix} 1 & y & \\ & 1 & \\ & & -2 \end{pmatrix} \quad \begin{pmatrix} 1 & y & \\ -y & 1 & \\ & & -2 \end{pmatrix} \quad \begin{pmatrix} 1 & & \\ & y & \\ & & -1-y \end{pmatrix}$$

Computing centralizers shows that ϕ is conjugate to ϕ_4 , ϕ_5 , or ϕ_6 , respectively. (If $\phi(e_1)$ is diagonal with two identical diagonal entries, then some element of its centralizer isn't diagonal with two identical diagonal entries but still has one of the above forms.)

To show that none of these are conjugate to each other, observe that ϕ_4 , ϕ_5 , and ϕ_6 each send $(2, 1) \in \mathbb{R}^2$ to a matrix in Jordan form which is not the Jordan form of a matrix in the image of any other ϕ_i . \square

The following definition is needed for Lemma 5.15's computation of cohomology; a survey can be found in [Wag10] or [AMR00, §2–4]⁶.

Definition 5.14 (Lie algebra cohomology, following [Wag10, §2]). Let M be a module of a Lie algebra \mathfrak{g} over a field k . The *Chevalley-Eilenberg complex* is the cochain complex

$$C^p(\mathfrak{g}, M) = \text{Hom}_k(\Lambda^p \mathfrak{g}, M)$$

with boundary maps

$$\begin{aligned} d_p : C^p &\rightarrow C^{p+1} \\ (d_p c)(x_1, \dots, x_{p+1}) &= \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1}) \\ &\quad + \sum_{1 \leq i \leq p+1} (-1)^{i+1} x_i c(x_1, \dots, \hat{x}_i, \dots, x_{p+1}) \end{aligned}$$

where \hat{x}_i means x_i should be omitted. The cohomology of \mathfrak{g} with coefficients in M is defined to be the cohomology of this complex and denoted $H^p(\mathfrak{g}, M)$.

Lemma 5.15. *If \mathbb{R}^2 acts on \mathbb{R}^3 via $\phi : \mathbb{R}^2 \rightarrow \mathfrak{sl}_3 \mathbb{R}$, then $H^2(\mathbb{R}^2; \mathbb{R}^3) \cong \mathbb{R}^3 / \phi(\mathbb{R}^2)(\mathbb{R}^3)$.*

Proof. All 3-cochains are zero since $\Lambda^3 \mathbb{R}^2 = 0$, so every 2-cochain is a cocycle. A 2-cochain is a map $\Lambda^2 \mathbb{R}^2 \rightarrow \mathbb{R}^3$, which can be recovered from the image of a spanning element $e_1 \wedge e_2 \in \Lambda^2 \mathbb{R}^2$; this identifies the 2-cocycles with \mathbb{R}^3 .

It then suffices to identify the 2-coboundaries with $\phi(\mathbb{R}^2)(\mathbb{R}^3)$. First, for all 1-cochains c ,

$$(dc)(x, y) = \phi(x)c(y) - \phi(y)c(x) \in \phi(\mathbb{R}^2)(\mathbb{R}^3).$$

⁶ An almost identical version, generalized to super (i.e. $\mathbb{Z}/2\mathbb{Z}$ -graded) Lie algebras, has been published as [AMR05].

Conversely, given $u \in \mathbb{R}^2$ and $w \in \mathbb{R}^3$, take $u' \in \mathbb{R}^2$ linearly independent from u and define a 1-cochain c by $c(u) = 0$ and $c(u') = w$. Then

$$(dc)(u, u') = \phi(u)c(u') = \phi(u)(w). \quad \square$$

The proof of Prop. 5.12—that only one extension of \mathbb{R}^2 by \mathbb{R}^3 produces a maximal geometry with trivial isotropy and nilradical \mathbb{R}^3 —is within reach, now that the above data can be used to describe the extensions in just enough detail to rule most of them out.

Proof of Prop. 5.12. Suppose $G = G/\{1\}$ is a maximal model geometry of dimension 5 and \mathfrak{g} is its Lie algebra, with $\text{nil } \mathfrak{g} \cong \mathbb{R}^3$. We have already established that \mathfrak{g} is an extension

$$0 \rightarrow \mathbb{R}^3 \rightarrow \mathfrak{g} \rightarrow \mathbb{R}^2 \rightarrow 0$$

where \mathbb{R}^2 acts faithfully and tracelessly by lifting to \mathfrak{g} and taking brackets (Prop. 5.8 and Lemma 5.11).

Step 1: Use $H^2(\mathbb{R}^2; \mathbb{R}^3)$ to classify extensions. For each of the actions $\phi : \mathbb{R}^2 \hookrightarrow \mathfrak{sl}_3\mathbb{R}$ (Lemma 5.13), we have by Lemma 5.15 that

$$H^2(\mathbb{R}^2; \mathbb{R}^3) \cong \mathbb{R}^3 / \phi(\mathbb{R}^2)(\mathbb{R}^3).$$

Explicitly, if ϕ_1, \dots, ϕ_6 name the actions in Lemma 5.13 and $\{e_1, e_2, e_3\}$ is the standard basis of \mathbb{R}^3 , then

$$H^2(\mathbb{R}^2; \mathbb{R}^3) \cong \begin{cases} \mathbb{R}e_3 & \text{if } \phi = \phi_1 \text{ or } \phi_2 \\ \mathbb{R}e_2 + \mathbb{R}e_3 & \text{if } \phi = \phi_3 \\ 0 & \text{if } \phi = \phi_4, \phi_5, \text{ or } \phi_6. \end{cases}$$

Isomorphism classes of extensions of \mathbb{R}^2 by \mathbb{R}^3 are in bijection with classes $[c] \in H^2(\mathbb{R}^2; \mathbb{R}^3)$ [AMR00, Cor. 9]; so the extension with ϕ_6 splits.

Moreover, defining relations for \mathfrak{g} can be recovered by using the cocycle as an \mathbb{R}^3 -valued bracket on \mathbb{R}^2 [AMR00, Eqn. 5.5]. Explicitly, identify \mathfrak{g} as vector space with $\mathbb{R}^3 \oplus \mathbb{R}^2$; and for $k_i \in \mathbb{R}^3$ and $q_i \in \mathbb{R}^2$, define

$$[(k_1, q_1), (k_2, q_2)] = (\phi(q_1)(k_2) - \phi(q_2)(k_1), 0).$$

Step 2: ϕ_1, \dots, ϕ_4 produce the wrong nilradical. Let $v = (0, 1) \in \mathbb{R}^2$, and let $\mathfrak{n} = \mathbb{R}^3 + \mathbb{R}v$. Then \mathfrak{n} is an ideal since $\mathfrak{g}^2 \subseteq \mathbb{R}^3 \subseteq \mathfrak{n}$.

Since v acts by a nilpotent matrix on \mathbb{R}^3 , using the distributive law to compute \mathfrak{n}^4 yields

$$\mathfrak{n}^4 = [v, [v, [v, \mathbb{R}^3]]] = 0.$$

Thus \mathfrak{n} is also nilpotent; so ϕ_1, \dots, ϕ_4 do not produce \mathfrak{g} where $\text{nil } \mathfrak{g} \cong \mathbb{R}^3$.

Step 3: ϕ_5 produces a non-maximal geometry. If $\phi = \phi_5$, then $G \cong (\mathbb{C} \times \mathbb{R}) \rtimes \mathbb{R}^2$, where $(x, y) \in \mathbb{R}^2$ acts as scaling by e^{x+iy} on \mathbb{C} and by e^{-2x} on \mathbb{R} . The action of $S^1 \subset \mathbb{C}$ on \mathbb{C} commutes with this, so $S^1 \subseteq \text{Aut}(G)$. Then $G/1 \cong G \rtimes S^1/S^1$. \square

The case $\phi = \phi_6$ has \mathbb{R}^2 acting on \mathbb{R}^3 by traceless diagonal matrices; the following section will show that it produces a model geometry.

5.3 The $\mathbb{R}^3 \rtimes \mathbb{R}^2$ geometry

When point stabilizers are 0-dimensional, a geometry admitting a finite-volume quotient by isometries is a Lie group admitting a lattice (Prop. 2.5). So to show this geometry is a model geometry, it suffices to prove the following.

Proposition 5.16. *The Lie group $\mathbb{R}^3 \rtimes \mathbb{R}^2$, where \mathbb{R}^2 acts on \mathbb{R}^3 as the diagonal matrices with determinant 1 and positive eigenvalues, admits a lattice.*

Proof. By applying linear changes of coordinates in \mathbb{R}^3 and \mathbb{R}^2 , it suffices to construct a group isomorphic to $\mathbb{Z}^3 \rtimes \mathbb{Z}^2$ where the action is by diagonalizable matrices with positive eigenvalues. One can use the ring of integers of a number field, acted on by its group of units.

If K is a cubic number field, then its ring of integers \mathcal{O}_K is isomorphic as a group to \mathbb{Z}^3 ; and if K is totally real, Dirichlet's unit theorem (see e.g. [Neu99, Thm. 7.4]) implies that the group of units \mathcal{O}_K^\times has rank $3 - 1 = 2$. So one may take $\mathcal{O}_K \rtimes 2U$ where $U \subseteq \mathcal{O}_K^\times$ is free abelian of rank 2, and $2U$ consists of the squares in U so that the action will have positive eigenvalues.

To obtain such a field, take $K = \mathbb{Q}[x]/(p(x))$ where $p \in \mathbb{Z}[x]$ is a monic irreducible cubic with three distinct real roots. \square

Example 5.17. Let $p(x) = 1 - 3x + x^3$, so $K \cong \mathbb{Q}[\alpha]$ where α is any root of p . Then

$$\begin{aligned}\alpha(3 - \alpha^2) &= 3\alpha - \alpha^3 = 1 \\ (1 - \alpha)(2 - \alpha - \alpha^2) &= 2 - 3\alpha + \alpha^3 = 1,\end{aligned}$$

so α and $1 - \alpha$ are units in $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_K$.

To prove that α and $1 - \alpha$ are independent in \mathcal{O}_K^\times , let $v_1, v_2, v_3 \in K \otimes_{\mathbb{Q}} \mathbb{R}$ be a basis of eigenvectors for multiplication by α on K as a \mathbb{Q} -vector space. Let $U \subseteq \mathcal{O}_K^\times$ be the subgroup generated by α and $1 - \alpha$, and define a homomorphism

$$\begin{aligned}\phi : U &\rightarrow \mathbb{R}^3 \\ a &\mapsto (\log |\lambda_1|, \log |\lambda_2|, \log |\lambda_3|) \text{ where } av_i = \lambda_i v_i.\end{aligned}$$

Then $\phi(\alpha) \approx (0.6, -1, 0.4)$ and $\phi(1 - \alpha) \approx (1, -0.4, -0.6)$ are linearly independent since they lie in non-opposite octants; so $U \cong \phi(U) \cong \mathbb{Z}^2$.

The matrices by which \mathbb{Z}^2 acts on \mathbb{Z}^3 are products of even powers of α and $1 - \alpha$, expressed in the basis $(1, \alpha, \alpha^2)$. That is, the action $\mathbb{Z}^2 \rightarrow \text{Aut } \mathbb{Z}^3 = \text{SL}(3, \mathbb{Z})$ is given by

$$x, y \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}^{2x} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -3 \\ 0 & -1 & 1 \end{pmatrix}^{2y}.$$

5.4 Classification of semidirect sums with \mathbb{R}

Prop. 5.2 has reduced the discovery of candidates to a classification of semidirect products—that is, classifying semidirect sums $\mathfrak{n} \rtimes \mathbb{R}$ where \mathfrak{n} is nilpotent of dimension 4. The program followed here is to classify such \mathfrak{n} , determine possible actions of \mathbb{R} on each \mathfrak{n} , and determine which semidirect sums produce model geometries (i.e. are tangent to Lie groups admitting lattices).

Proposition 5.18 (see also [PSWZ76, Table I]). *Any nilpotent Lie algebra \mathfrak{n} of dimension 4 is isomorphic to \mathbb{R}^4 or $\mathbb{R} \oplus \mathfrak{n}_3$ or \mathfrak{n}_4 .*

Proof. The dimension of $\mathfrak{n}^2 = [\mathfrak{n}, \mathfrak{n}]$ distinguishes the three Lie algebras above, so use this dimension to determine \mathfrak{n} .

- $\mathfrak{n}^2 \neq \mathfrak{n}$ since \mathfrak{n} is nilpotent.
- If $\dim \mathfrak{n}^2 = 3$, then pick $x \in \mathfrak{n} \setminus \mathfrak{n}^2$. Then

$$\mathfrak{n}^2 = (\mathbb{R}x + \mathfrak{n}^2)^2 = [\mathbb{R}x, \mathfrak{n}^2] + [\mathfrak{n}^2, \mathfrak{n}^2] \subseteq \mathfrak{n}^3,$$

so the lower central series stabilizes at $\mathfrak{n}^2 \neq 0$, so this never occurs for nilpotent \mathfrak{n} .

- If $\dim \mathfrak{n}^2 = 2$, then $\dim \mathfrak{n}^3$ is either 0 or 1.
 - If $\dim \mathfrak{n}^3 = 0$, then \mathfrak{n}^2 is central. Pick nonzero:

$$\begin{aligned} x_3 &\notin \mathfrak{n}^2 \\ x_2 &\notin \mathfrak{n}^2 + \mathbb{R}x_3 \\ x_1 &\in \mathfrak{n}^2 \text{ such that } [x_3, x_2] = 0 \text{ or } x_1 \\ y &\in \mathfrak{n}^2 \text{ completing these to a basis.} \end{aligned}$$

Then \mathfrak{n} is either \mathbb{R}^4 or $\mathbb{R} \oplus \mathfrak{n}_3$. (In fact, for both of these, \mathfrak{n}^2 is too small.)

- If $\dim \mathfrak{n}^3 = 1$, then choose nonzero:

$$\begin{aligned} x_1 &\in \mathfrak{n}^3 \\ x_2 &\in \mathfrak{n}^2 \setminus \mathfrak{n}^3 \\ x_4 &\in \mathfrak{n} \setminus \mathfrak{n}^2 \text{ such that } [x_4, x_2] = x_1 \\ x_3 &\in \mathfrak{n} \setminus \mathfrak{n}^2 \text{ such that } [x_4, x_3] = x_2. \end{aligned}$$

Since $[x_3, x_2] \in \mathfrak{n}^3$, it could be any multiple of x_1 . Replacing x_3 by an element of $x_3 + \mathbb{R}x_4$ to make its bracket with x_2 zero yields a basis demonstrating $\mathfrak{n} \cong \mathfrak{n}_4$.

- If $\dim \mathfrak{n}^2 = 1$, then \mathfrak{n}^2 is central, as the last term in the lower central series. For any linear complement V of \mathfrak{n}^2 , the Lie bracket induces $V \wedge V \rightarrow \mathfrak{n}^2 \cong \mathbb{R}$, which is necessarily degenerate since V has odd dimension. Then we can choose x_1 spanning \mathfrak{n}^2 , x_2 and x_3 such that $[x_3, x_2] = x_1$, and y demonstrating the degeneracy of $V \wedge V \rightarrow \mathfrak{n}^2$. This basis demonstrates $\mathfrak{n} \cong \mathbb{R} \oplus \mathfrak{n}_3$.
- If $\dim \mathfrak{n}^2 = 0$, then $\mathfrak{n} \cong \mathbb{R}^4$.

□

5.4.1 \mathbb{R}^4 semidirect sums

Suppose G is a model geometry and $\mathfrak{g} = \mathfrak{n} \ltimes_{\phi} \mathbb{R}$. If $\mathfrak{n} = \mathbb{R}^4$, a linear change of coordinates puts the image of 1 under $\phi : \mathbb{R} \rightarrow \mathfrak{sl}_4 \mathbb{R}$ in Jordan form. Listing Jordan forms, grouped by number of blocks, yields the following. (Omitted entries are zero, and “*” entries are subject only to the restriction

that the whole matrix is traceless.)

$$\begin{aligned}
\phi_1(1) &= \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} & \phi_{1'}(1) &= \begin{pmatrix} & \lambda & 1 & \\ -\lambda & & & 1 \\ & & -\lambda & \\ & & & \lambda \end{pmatrix} \\
\phi_2(1) &= \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & -\lambda & 1 \\ & & & -\lambda \end{pmatrix} & \phi_{2'}(1) &= \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & \\ & & & -3\lambda \end{pmatrix} \\
\phi_{2''}(1) &= \begin{pmatrix} \lambda & \mu & & \\ -\mu & \lambda & & \\ & & -\lambda & 1 \\ & & & -\lambda \end{pmatrix} & \phi_{2'''}(1) &= \begin{pmatrix} \lambda & \mu_1 & & \\ -\mu_1 & \lambda & & \\ & & -\lambda & \mu_2 \\ & & -\mu_2 & -\lambda \end{pmatrix} \\
\phi_3(1) &= \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & * & \\ & & & * \end{pmatrix} & \phi_{3'}(1) &= \begin{pmatrix} \lambda & \mu & & \\ -\mu & \lambda & & \\ & & * & \\ & & & * \end{pmatrix} \\
\phi_4(1) &= \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix}
\end{aligned}$$

With this, it becomes possible to classify maximal model geometries of the form $\mathbb{R}^4 \rtimes \mathbb{R}$; computing characteristic polynomials recovers the list in Thm. 1.1(ii)(b) from the following statement.

Proposition 5.19 (Classification of $\mathbb{R}^4 \rtimes \mathbb{R}$ geometries).

- (i) If $G = G/\{1\}$ is a maximal model geometry with Lie algebra $\mathfrak{g} = \mathbb{R}^4 \rtimes_{\phi} \mathbb{R}$, then ϕ can be taken to be one of ϕ_1 , ϕ_2 ($\lambda = 1$), $\phi_{2'}$ ($\lambda = 0$), ϕ_3 ($\lambda = 0$), or ϕ_4 .
- (ii) All cases listed in (i) are model geometries, except that the group with Lie algebra $\mathbb{R}^4 \rtimes_{\phi_4} \mathbb{R}$ is a model geometry if $\exp \phi_4(t)$ has integer characteristic polynomial for some $t \neq 0$.

Note that some of the actions ϕ_i depend on parameters, and not all of the parameter values produce model geometries. Fortunately, there is an easy-to-state necessary and sufficient condition for model geometries arising in this case: recall that G/G_p is a model geometry if and only if some lattice $\Gamma \subset G$ intersects no conjugate of G_p nontrivially (Prop. 2.5(ii)); so in the case of $G_p = \{1\}$, this reduces to the question of whether a lattice exists, which is determined by the following condition.

Lemma 5.20 ([Fil83, Cor. 6.4.3]). *A unimodular, non-nilpotent $\mathbb{R}^n \rtimes_{\exp \phi} \mathbb{R}$ admits a lattice if and only if there is $0 \neq t \in \mathbb{R}$ such that the characteristic polynomial of $\exp \phi(t)$ has coefficients in \mathbb{Z} .*

Proof of Prop. 5.19. The complete list of actions of \mathbb{R} on \mathbb{R}^4 would produce a long list of cases, so a first step will be to eliminate actions producing non-maximal geometries. Each remaining group is then examined to determine whether it admits a lattice. The cases are grouped by the number of Jordan blocks.

Preparatory step: ignore non-maximal geometries. If $\phi(1)$ has non-real eigenvalues, then it (and the 1-parameter subgroup of $\mathrm{SL}(4, \mathbb{R})$ it generates) commutes with rotations on some 2-dimensional eigenspace. These rotations form an $S^1 \subseteq \mathrm{Aut}(G)$; so $G/\{1\}$ is not maximal, since it is subsumed by $G \rtimes S^1/S^1$. This eliminates $\phi_{1'}$, $\phi_{2''}$, $\phi_{2'''}$, and $\phi_{3'}$.

Case 1: $\phi(1)$ has 1 Jordan block. Then $\phi = \phi_1$. Since

$$\exp \phi_1(6) = \begin{pmatrix} 1 & 6 & 18 & 36 \\ & 1 & 6 & 18 \\ & & 1 & 6 \\ & & & 1 \end{pmatrix}$$

has integer entries, its characteristic polynomial has integer coefficients; so the resulting $\mathbb{R}^4 \rtimes \mathbb{R}$ admits a lattice by Lemma 5.20 above. In fact the lattice can be taken to be a subgroup isomorphic to $\mathbb{Z}^4 \rtimes_{\exp \phi_1(6)} \mathbb{Z}$, with fundamental domain $\{((\exp \phi_1(t))x, t) \mid t \in (0, 6), x \in [0, 1]^4\}$.

Case 2a: $\phi = \phi_2$ (2 Jordan blocks). If $\lambda = 0$ then $\phi_2(t)$ commutes with an S^1 of rotations (it coincides with the $\lambda = 0$ case for $\phi_{1'}$) and will not produce a maximal geometry.

If instead $\lambda \neq 0$, then by rescaling ϕ_2 and changing basis to put the new $\phi_2(1)$ in Jordan form, we can assume $\lambda = 1$, so this produces at most one new geometry. Exponentiating yields the 1-parameter subgroup

$$\left\{ \begin{pmatrix} e^t & te^t & & \\ & e^t & & \\ & & e^{-t} & te^{-t} \\ & & & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathrm{SL}(\mathbb{R}^4).$$

Reordering the basis elements turns this into the block matrices

$$\left\{ \begin{pmatrix} A(t) & tA(t) \\ & A(t) \end{pmatrix} : t \in \mathbb{R}, A(t) = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \right\} \subset \mathrm{SL}(\mathbb{R}^4).$$

If B diagonalizes $C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then after conjugating by the block diagonal matrix with diagonal

blocks B^{-1} , this 1-parameter subgroup contains an element of the form $\begin{pmatrix} C & sC \\ & C \end{pmatrix}$, with $s \in \mathbb{R}$.

After rescaling the first two basis elements, we conclude as in Case 1 above that the Lie group $\mathbb{R}^4 \rtimes_{\exp \phi_2} \mathbb{R}$ admits a lattice isomorphic to $\mathbb{Z}^4 \rtimes_{A'} \mathbb{Z}$, where

$$A' = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ & & 2 & 1 \\ & & 1 & 1 \end{pmatrix}.$$

Case 2b: $\phi = \phi_{2'}$ (2 Jordan blocks). We will prove in this case that the resulting $G = \mathbb{R}^4 \rtimes \mathbb{R}$ is a model geometry if and only if $\lambda = 0$.

By Lemma 5.20 above, if G is a model geometry then $\exp \phi_{2'}(t)$ has characteristic polynomial $p(x) \in \mathbb{Z}[x]$ for some $t \neq 0$. Then $e^{t\lambda}$ is a triple root of p , so its minimal polynomial over \mathbb{Q} divides p at least 3 times. Since $\deg p = 4$, the minimal polynomial of $e^{t\lambda}$ is linear; so $e^{t\lambda} \in \mathbb{Q}$. Since $e^{t\lambda}$ is a rational root of a polynomial whose first and last coefficients are 1, the rational root theorem implies $e^{t\lambda} = \pm 1$. Then since $t \neq 0$ and λ is real, $\lambda = 0$.

Conversely, $\lambda = 0$ then $\exp \phi_{2'}(2)$ has integer entries; so G admits the lattice $\mathbb{Z}^4 \rtimes_{\exp \phi_{2'}(2)} \mathbb{Z}$. In the notation of [Fil83, §6.4.6], this gives the geometry $G_3 \times \mathbb{R}$.

Case 3: $\phi = \phi_3$ (3 Jordan blocks). Again we prove that the resulting $\mathbb{R}^4 \rtimes \mathbb{R}$ is a model geometry if and only if $\lambda = 0$; moreover, this produces only one geometry.

If a model geometry results, then $\exp \phi_3(t)$ has characteristic polynomial $p(x) \in \mathbb{Z}[x]$ for some $t \neq 0$. Then $e^{t\lambda}$ is a double root of $p(x)$, so its minimal polynomial over \mathbb{Q} divides $p(x)$ at least twice. If $e^{t\lambda} \notin \mathbb{Q}$, then the two eigenvalues of $\phi_3(1)$ that aren't λ are identical; so the last two coordinates admit an S^1 of rotations, making G non-maximal. Then $e^{t\lambda} \in \mathbb{Q}$; so as in Case 2b, $\lambda = 0$ by the rational root theorem.

If $\lambda = 0$, then ϕ_3 and the first basis vector of \mathbb{R}^4 can be rescaled so that

$$\phi_3(1) = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & \alpha & \\ & & & -\alpha \end{pmatrix}$$

where $\alpha = \ln \frac{3+\sqrt{5}}{2}$. (If the diagonal entries were all zero, then the last two coordinates would admit an S^1 of rotations.) Exponentiating this yields a 1-parameter subgroup of $\mathrm{SL}(\mathbb{R}^4)$ containing a matrix A that in some basis becomes

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 2 & 1 \\ & & 1 & 1 \end{pmatrix},$$

so G admits a lattice isomorphic to $\mathbb{Z}^4 \rtimes_A \mathbb{Z}$.

Case 4: $\phi = \phi_4$ (4 Jordan blocks). The criterion claimed in (ii) for producing a model geometry is merely Lemma 5.20 above. This completes the proof of Prop. 5.19. \square

Remark 5.21 (Alternative parametrization of the $\phi = \phi_4$ case). The 3-dimensional family of maps ϕ_4 produces a 2-dimensional family of groups $\mathbb{R}^4 \rtimes_{\exp \phi_4} \mathbb{R}$ (due to the ability to rescale the \mathbb{R} factor), but not all are model geometries—by Lemma 5.20 above, $\exp \phi_4(t)$ has to have an integer characteristic polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + 1 \in \mathbb{Z}[x]$ for some nonzero t . Allowing for rescaling the \mathbb{R} factor, one could instead parametrize this family of geometries by the coefficients $(a, b, c) \in \mathbb{Z}^3$; but the correspondence is not a bijection due to the following.

- (i) Since $\phi_4(t)$ has 4 real eigenvalues, $p(x)$ must have 4 nonnegative real roots.
- (ii) Since duplicate eigenvalues of $\phi_4(t)$ would make $\exp \phi_4(t)$ commute with an S^1 of rotations, the roots of $p(x)$ must be distinct if it comes from a maximal geometry.
- (iii) If $\exp \phi_4(t)$ has integer characteristic polynomial (i.e. is an integer matrix in some basis), then so does $\exp \phi_4(nt)$ for any $n \in \mathbb{Z}$; so each geometry corresponds to a \mathbb{Z} -family of polynomials.

In [Fil83, §6.4.8], Filipkiewicz gave a detailed description of this kind of parametrization for the analogous family $\mathrm{Sol}_{m,n}^4$ in dimension 4.

5.4.2 \mathfrak{n}_4 semidirect sums

The classification of $\mathfrak{n}_4 \rtimes \mathbb{R}$ geometries has a slightly different flavor from the $\mathbb{R}^4 \rtimes \mathbb{R}$ case—there are fewer derivations of \mathfrak{n}_4 than of \mathbb{R}^4 ; but the tradeoff is that they require a bit more work to find. The result is the following.

Proposition 5.22 (Classification of $\mathfrak{n}_4 \rtimes \mathbb{R}$ geometries). *Suppose $G = G/\{1\}$ is a model geometry whose Lie algebra \mathfrak{g} is of the form $\mathfrak{n}_4 \rtimes \mathbb{R}$ and not expressible as $\mathbb{R}^4 \rtimes \mathbb{R}$. Then \mathfrak{g} has basis $\{x_1, x_2, x_3, x_4, x_5\}$ with*

$$[x_4, x_2] = x_1 \quad [x_4, x_3] = x_2 \quad [x_5, x_3] = x_1 \quad [x_5, x_4] = 0 \text{ or } x_3,$$

and all other brackets not determined by skew-symmetry are zero. Both Lie algebras thus described are the Lie algebras of model geometries.

Most of the proof lies in describing a derivation D that makes $\mathfrak{n}_4 \rtimes_D \mathbb{R}$ the Lie algebra of a model geometry. Predictably, this case begins with the computation of the traceless outer derivation algebra $\text{sout}(\mathfrak{n}_4)$ (Lemma 5.23). This will be followed by an extra condition on D for model geometries (Lemma 5.24) and the proof of Prop. 5.22.

Lemma 5.23. *Every element of $\text{sout}(\mathfrak{n}_4)$ is represented by a matrix of the form*

$$D_{4,a,b,c} = \begin{pmatrix} 2a & 0 & b & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -4a & c \\ 0 & 0 & 0 & 3a \end{pmatrix},$$

with respect to the basis in Definition 5.4.

Proof. Suppose $D \in \text{der } \mathfrak{n}_4$ is traceless. $\text{ad } \mathfrak{n}_4$ consists of the matrices

$$\begin{pmatrix} 0 & c & 0 & a \\ 0 & 0 & c & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Any derivation D is upper triangular since it preserves the following filtration by characteristic ideals.

$$\begin{aligned} \text{span}(x_1) &= Z(\mathfrak{n}) \\ \text{span}(x_1, x_2) &= \mathfrak{n}_4^2 \\ \text{span}(x_1, x_2, x_3) &= \{x \in \mathfrak{n}_4 \mid \dim[x, \mathfrak{n}_4] < 2\} \end{aligned}$$

Hence, up to an inner derivation, we may write D as

$$\begin{aligned} Dx_1 &= a_1 x_1 \\ Dx_2 &= b_2 x_2 \\ Dx_3 &= c_1 x_1 + c_2 x_2 + c_3 x_3 \\ Dx_4 &= d_3 x_3 + d_4 x_4. \end{aligned}$$

Using the Leibniz rule,

$$\begin{aligned} a_1 x_1 &= D x_1 = [D x_4, x_2] + [x_4, D x_2] = (d_4 + b_2) x_1 \\ b_2 x_2 &= D x_2 = [D x_4, x_3] + [x_4, D x_3] = (d_4 + c_3) x_2 + c_2 x_1, \end{aligned}$$

so $c_2 = 0$ and $c_3 = b_2 - d_4 = a_1 - 2d_4$. Finally, if D is traceless,

$$0 = a_1 + b_2 + c_3 + d_4 = 3c_3 + 4d_4. \quad \square$$

Lemma 5.24. *Suppose $G = G/\{1\}$ is a model geometry whose Lie algebra \mathfrak{g} is of the form $\mathfrak{n} \rtimes \mathbb{R}$ where \mathfrak{n} is nilpotent. Then \mathbb{R} acts tracelessly on $Z(\mathfrak{n})$.*

Proof. Since $Z(\mathfrak{n})$ is a characteristic ideal, it is stable under the action of \mathbb{R} . Suppose \mathbb{R} acts with nonzero trace on $Z(\mathfrak{n})$.

Then \mathbb{R} acts non-nilpotently, so $\mathfrak{n} = \text{nil } \mathfrak{g}$. Write $G = N \rtimes \mathbb{R}$ where N is the simply-connected Lie group with Lie algebra \mathfrak{n} .

Since G is a model geometry, it admits a lattice Γ . The nilradical of a solvable Lie group inherits lattices [Mos71, Lemma 3.9]—that is, $N \cap \Gamma$ is a lattice in N , and $\Gamma/(N \cap \Gamma)$ is a lattice in \mathbb{R} . The latter implies that some element of Γ acts by conjugation with determinant > 1 on $Z(N)$.

Since $Z(N)$ is a term in the upper central series of N , the intersection $Z(N) \cap \Gamma$ is a lattice in $Z(N)$ [Rag72, Prop. 2.17]. This is impossible since no lattice of \mathbb{R}^k is stable under a linear map with determinant > 1 . \square

Given the above, the classification of model geometries with tangent algebra of the form $\mathfrak{n}_4 \rtimes \mathbb{R}$ is within reach and will recover the list in Thm. 1.1(ii)(c).

Proof of Prop. 5.22. The action of \mathbb{R} on \mathfrak{n}_4 is represented by a derivation of the form described above in Lemma 5.23. The following steps recover the Lie algebra defined in the statement of Prop. 5.22.

1. Since G needs to be a model geometry, $a = 0$ by the tracelessness condition from Lemma 5.24.
2. If $b = 0$, then x_1, x_2, x_3 , and x_5 span an abelian ideal, so G is also expressible as $\mathbb{R}^4 \rtimes \mathbb{R}$. Hence $b \neq 0$. Rescale x_5 by a factor of b^{-1} —i.e. replace x_5 with $b^{-1}x_5$ —to make $b = 1$.
3. If $c \neq 0$, then similarly rescale $\text{span}(x_1, x_2, x_3)$ by a factor of c to make $c = 1$.

In the end only one parameter can vary— c may be 0 or 1. Either way, \mathfrak{g} is nilpotent and expressed in a basis with integral structure constants; so the corresponding simply-connected group G admits a lattice [Rag72, Thm. 2.12]. Therefore G is a model geometry. \square

5.4.3 $\mathbb{R} \oplus \mathfrak{n}_3$ semidirect sums

Despite only producing two geometries, this case is the messiest of the three— $\mathbb{R} \oplus \mathfrak{n}_3$ has enough outer derivations to make this a long problem, yet not so many that a systematic approach is as easy as listing Jordan blocks in $\mathfrak{sl}_4 \mathbb{R}$. The main result is the following.

Proposition 5.25 (Classification of $(\mathbb{R} \times \text{Heis}_3) \rtimes \mathbb{R}$ geometries). *Suppose $G = G/\{1\}$ is a maximal model geometry with Lie algebra \mathfrak{g} of the form $(\mathbb{R} \oplus \mathfrak{n}_3) \rtimes_D \mathbb{R}$. If \mathfrak{g} contains no ideal isomorphic to \mathbb{R}^4 or \mathfrak{n}_4 , then \mathfrak{g} is isomorphic to a Lie algebra constructed by letting $1 \in \mathbb{R}$ act on $\mathbb{R} \oplus \mathfrak{n}_3 = \text{span}(y, x_1, x_2, x_3)$ by*

$$\begin{aligned} x_2 &\mapsto x_2 & x_3 &\mapsto -x_3 & y &\mapsto 0 \text{ or } x_1. \end{aligned}$$

For both Lie algebras thus named, the corresponding simply-connected group G is a model geometry.

The strategy will broadly be the same as before: list elements of $\text{sout}(\mathbb{R} \oplus \mathfrak{n}_3)$ by which $1 \in \mathbb{R}$ can act; use coordinate changes to cluster the resulting Lie algebras by isomorphism type; and show that each admits a lattice. To save some work, we begin with a lemma that eliminates some of $\text{sout}(\mathbb{R} \oplus \mathfrak{n}_3)$ from consideration.

Lemma 5.26. *Let $\mathfrak{n} = \mathbb{R} \oplus \mathfrak{n}_3$. If $G = G/\{1\}$ is a model geometry whose Lie algebra \mathfrak{g} is of the form $\mathfrak{n} \rtimes \mathbb{R}$, then \mathbb{R} acts tracelessly on $\mathfrak{n}^2 = Z(\mathfrak{n}_3)$.*

Proof. Suppose \mathbb{R} acts with nonzero trace on the characteristic ideal $Z(\mathfrak{n}_3)$. Let Γ be a lattice in $G = (\mathbb{R} \times \text{Heis}_3) \rtimes \mathbb{R}$. Following the argument in Lemma 5.24, some element of Γ acts with determinant < 1 on $Z(\text{Heis}_3)$, and $\Gamma \cap (\mathbb{R} \times \text{Heis}_3)$ maps to a lattice in $(\mathbb{R} \times \text{Heis}_3)/Z(\mathbb{R} \times \text{Heis}_3) \cong \mathbb{R}^2$.

Then over some two linearly independent elements of \mathbb{R}^2 lie two elements of Γ ; their commutator is a nontrivial element of $Z(\text{Heis}_3) \cong \mathbb{R}$. This is impossible since no discrete subgroup of \mathbb{R} is stable under an automorphism with determinant < 1 . \square

In combination with the requirement that \mathbb{R} acts tracelessly on $Z(\mathfrak{n})$ (Lemma 5.24), this brings the dimension of the relevant subspace of $\text{sout}(\mathbb{R} \oplus \mathfrak{n}_3)$ down to 6.

Lemma 5.27. *If $\mathfrak{n} = \mathbb{R} \oplus \mathfrak{n}_3$ has ordered basis (y, x_1, x_2, x_3) (with $[x_3, x_2] = x_1$ as in Definition 5.4), then every element of $\text{sout}(\mathfrak{n})$ which also acts tracelessly on $\mathfrak{n}^2 = \text{span}(x_1)$ and $Z(\mathfrak{n}) = \text{span}(x_1, y)$ is represented by a matrix of the form*

$$\begin{pmatrix} 0 & 0 & e & f \\ d & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{pmatrix}.$$

Proof. The tracelessness assumptions, along with the fact that derivations preserve characteristic ideals, imply that $Dx_1 = 0$, that $Dy \in \text{span}(x_1)$, and that the last two diagonal entries add to 0.

By subtracting multiples of the inner derivations $\text{ad } x_2$ and $\text{ad } x_3$, we can assume that Dx_2 and Dx_3 have zero x_1 coordinate. \square

The proof of Prop. 5.25 which now follows is similar in spirit to that for \mathfrak{n}_4 (Prop. 5.22)—it shows that most of this space of derivations either produces no maximal model geometries or produces geometries accounted for by previous cases.

Proof of Prop. 5.25. Give \mathfrak{g} the ordered basis (y, x_1, x_2, x_3, z) , where the first four elements are as above in Lemma 5.27 and z spans the last \mathbb{R} factor. Let $D \in \text{der}(\mathbb{R} \oplus \mathfrak{n}_3)$ be the derivation by which z acts; and let D' be the lower right 2×2 block of D .

Preparatory step: Simplify D with coordinate changes. By changing basis in $\text{span}(x_2, x_3)$, we can assume D' is in Jordan form. By rescaling z , we may assume this block contains only 1, 0, and -1 as entries.

For $r \neq 0$, let $\mu_r : \mathbb{R} \oplus \mathfrak{n}_3 \rightarrow \mathbb{R} \oplus \mathfrak{n}_3$ be the automorphism given by

$$x_1 \mapsto r^2 x_1 \quad x_2 \mapsto r x_2 \quad x_3 \mapsto r x_3 \quad y \mapsto r y.$$

In the notation of Lemma 5.27, conjugating D by μ_r replaces d by rd while leaving the other entries unchanged; so we may also assume that d will always be 0 or 1.

Case 1: If $D' = 0$ then no new maximal geometries arise. In this case, D' commutes with the automorphisms of $\mathbb{R} \oplus \mathfrak{n}_3$ acting as rotations on $\text{span}(x_2, x_3)$. In the notation of Lemma 5.27, if $e = f = 0$, then these automorphisms also commute with D , so they extend to automorphisms of \mathfrak{g} . This would make G nonmaximal, since $G \cong G \rtimes S^1/S^1$.

Thus e and f are not both zero; so we can conjugate by a rotation to make $f = 0$ and a rescaling of y to make $e = 1$. Under this modified basis, D has the matrix

$$\begin{pmatrix} 0 & 1 & & \\ d & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

where d is 0 or 1. In either case, \mathfrak{g} has an ideal isomorphic to \mathbb{R}^4 or \mathfrak{n}_4 :

- If $d = 0$, then $\text{span}(y, x_1, x_3, z) \subset \mathfrak{g}$ is a 4-dimensional abelian ideal.
- If $d = 1$, then $\text{span}(x_1, y, x_2, z) \subset \mathfrak{g}$ is an ideal isomorphic to \mathfrak{n}_4 , with the basis ordered as in Definition 5.4—that is, the nonzero brackets are

$$[z, y] = x_1 \qquad [z, x_2] = y.$$

Case 2: If $D' \neq 0$ is skew-symmetric then G is non-maximal. As in Case 1, D' commuting with a compact group of automorphisms leads to the conclusion that D has matrix

$$\begin{pmatrix} 0 & 1 & & \\ d & 0 & & \\ & & 1 & \\ & & -1 & \end{pmatrix}.$$

Define

$$u_1 = x_1 \qquad u_2 = x_2 \qquad u_3 = x_3 - y \qquad v = y \qquad w = z - dx_2.$$

Then $\text{span}(v, u_1, u_2, u_3) \cong \mathbb{R} \oplus \mathfrak{n}_3$, and w acts in this ordered basis by the matrix

$$\begin{pmatrix} 0 & & & \\ d & 0 & & \\ & & 1 & \\ & & -1 & \end{pmatrix},$$

which commutes with a compact group of automorphisms, implying as in Case 1 that G is non-maximal.

Case 3: If D' is a single Jordan block then no new maximal geometries arise. In the notation of Lemma 5.27, either $e = 0$, or we can replace x_3 by $x_3 - \frac{f}{e}x_2$ to make $f = 0$. If one of e and f remains nonzero, then y can be rescaled to make it equal 1. This produces three matrices for D :

$$D_{1,d,0} = \begin{pmatrix} 0 & & & \\ d & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \qquad D_{1,d,1} = \begin{pmatrix} 0 & 1 & & \\ d & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \qquad D_{1,d,2} = \begin{pmatrix} 0 & & 1 \\ d & 0 & \\ & 0 & 1 \\ & & & 0 \end{pmatrix}$$

In all but one case, an ideal isomorphic to \mathfrak{n}_4 can be named by an ordered basis (v_0, v_1, v_2, v_3) where the nonzero brackets are $[v_3, v_2] = v_1$ and $[v_3, v_1] = v_0$.

$$\begin{aligned} D_{1,d,0} &: x_1, x_2, -z, x_3 \\ D_{1,1,1} &: x_1, y, x_2, z \\ D_{1,d,2} &: x_1, x_2 + y, dx_3 - z, x_3 \end{aligned}$$

If $D = D_{1,0,1}$, then \mathfrak{g} has basis $\{y, x_1, x_2, x_3, z\}$ with brackets

$$[z, x_2] = y \quad [x_3, x_2] = x_1 \quad [z, x_3] = x_2.$$

In this case \mathfrak{g} admits a compact group of automorphisms ϕ_θ defined by

$$\begin{aligned} z &\mapsto z \cos \theta + x_3 \sin \theta & y &\mapsto y \cos \theta + x_1 \sin \theta & x_2 &\mapsto x_2 \\ x_3 &\mapsto -z \sin \theta + x_3 \cos \theta & x_1 &\mapsto -y \sin \theta + x_1 \cos \theta, \end{aligned}$$

so $G/\{1\}$ is non-maximal when $D = D_{1,0,1}$.

Case 4: If $D' \neq 0$ is diagonal then \mathfrak{g} is determined up to isomorphism. In this case, D is the matrix

$$\begin{pmatrix} 0 & e & f \\ d & 0 & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Let

$$u_1 = x_1 \quad u_2 = x_2 + ey \quad u_3 = x_3 - fy \quad v = y \quad w = z - dex_3 - dfx_2.$$

Then u_1 is central, v commutes with all u_i , and

$$\begin{aligned} [u_3, u_2] &= u_1 \\ [w, u_2] &= u_2 \\ [w, u_3] &= -u_3 \\ [w, v] &= du_1. \end{aligned}$$

This shows that e and f do not affect the isomorphism type of \mathfrak{g} ; taking $e = f = 0$ recovers the definition in the statement of Prop. 5.25.

Final step: Verify these are model geometries A finite-volume quotient of G by a subgroup of its isometry group is a quotient by a lattice (Prop. 2.5(ii)); so it suffices to find a lattice in G .

Suppose $s \in \mathbb{R}$ is nonzero. Putting coordinates (y, x_1, x_2, x_3) on $\mathbb{R} \times \text{Heis}_3$, the semidirect product $(\mathbb{R} \times \text{Heis}_3) \rtimes_{e^{tA}} \mathbb{R}$ where

$$A = \begin{pmatrix} 0 & & & \\ s^{-1}d & 0 & & \\ & & -1 & 2 \\ & & 2 & 1 \end{pmatrix}$$

is, by the preparatory step, isomorphic to G . In particular, if $s = \frac{\ln \frac{3+\sqrt{5}}{2}}{\ln \sqrt{5}}$, then

$$e^{sA} = \begin{pmatrix} 1 & & & \\ d & 1 & & \\ & & 1 & 1 \\ & & 1 & 2 \end{pmatrix}.$$

This is an integer matrix, so it preserves the lattice Γ in $\mathbb{R} \times \text{Heis}_3$ consisting of the integer points. Then $\Gamma \rtimes_{e^{sA}} \mathbb{Z}$ is a lattice in G , with closed fundamental domain the standard unit cube of coordinates $[0, 1]^5$. \square

5.5 Maximality and distinctness

This section proves that the geometries $G = G/\{1\}$ obtained above are both maximal and distinct. Distinctness (Prop. 5.31) mostly uses dimensions of characteristic subalgebras to distinguish the corresponding Lie algebras from each other, and is summarized in Figure 1.3. Maximality (Prop. 5.30) uses the following theorem by Gordon and Wilson.⁷

Theorem 5.28 (part of [GW88, Thm. 4.3]). *Suppose G is a connected unimodular solvable Lie group with Lie algebra \mathfrak{g} . If the elements of $\text{ad } \mathfrak{g}$ have only real eigenvalues (i.e. \mathfrak{g} has all real roots), then in any invariant metric, there is some $C \subseteq \text{Aut } G$ such that $(\text{Isom } G)^0 = G \rtimes C$.*

Then $G \cong G \rtimes C/C$, and C becomes the stabilizer of the identity in G ; so for a maximal geometry, C is a maximal compact subgroup of $\text{Aut } G$. The following computation helps to find the maximal compact subgroups.

Lemma 5.29. *Suppose $H \subseteq \text{GL}(n, \mathbb{R})$ is a Lie group of consisting of block upper-triangular matrices, where k_1, \dots, k_m are the block sizes. Then its maximal compact subgroup is conjugate in $\text{GL}(n, \mathbb{R})$ to a subgroup of $\text{SO}(k_1) \times \dots \times \text{SO}(k_m)$.*

Proof. Let $G(k_1, \dots, k_m)$ be the group of block upper-triangular matrices where k_1, \dots, k_m are the block sizes. Since the maximal compact subgroup of H is a compact subgroup of $G(k_1, \dots, k_m)$, it suffices to compute the maximal compact subgroup of $G(k_1, \dots, k_m)$.

The determinant of a block upper-triangular matrix is the product of the determinants of the blocks. The straight-line homotopy that sends matrix entries outside the blocks to zero also preserves the determinant, thereby specifying a deformation retract of $G(k_1, \dots, k_m)$ onto the corresponding group of block diagonal matrices, which deformation retracts onto $\text{SO}(k_1) \times \dots \times \text{SO}(k_m)$. \square

Proposition 5.30. *Let G be the simply-connected Lie group with one of the following solvable Lie algebras \mathfrak{g} . Then $G/\{1\}$ is a maximal geometry.*

- (nilpotent) $\mathbb{R}^4 \rtimes \mathbb{R}$ where $1 \in \mathbb{R}$ acts with Jordan blocks with characteristic polynomials (x^4) or (x^3, x) .
- (nilpotent) $\mathfrak{n}_4 \rtimes \mathbb{R}$ where $1 \in \mathbb{R}$ acts by $x_3 \mapsto x_1$ and $x_4 \mapsto 0$ or x_3 (as named in Prop. 5.22).
- $\mathbb{R}^3 \rtimes \mathbb{R}^2$ where \mathbb{R}^2 acts by traceless diagonal matrices.
- $\mathbb{R}^4 \rtimes \mathbb{R}$ where $1 \in \mathbb{R}$ acts diagonalizably with distinct eigenvalues or with Jordan blocks with characteristic polynomials $(x^2, x-1, x+1)$ or $((x-1)^2, (x+1)^2)$.

⁷ This generalizes [Wil82, Thm. 2(3)], which is the same result for nilpotent Lie groups.

- $(\mathbb{R} \oplus \mathfrak{n}_3) \rtimes \mathbb{R}$ (as named in Prop. 5.25) where $1 \in \mathbb{R}$ acts by

$$x_2 \mapsto x_2 \qquad x_3 \mapsto -x_3 \qquad y \mapsto 0 \text{ or } x_1.$$

Proof. Using Thm. 5.28 (the Gordon-Wilson result) requires showing that \mathfrak{g} is unimodular—which is already known since G must admit a lattice to be a geometry—and that the eigenvalues of its adjoint representation are all real.

Case 1: G is nilpotent. For a nilpotent \mathfrak{g} , the adjoint representation's eigenvalues are all zero; so showing that the maximal compact connected subgroup of $\text{Aut } G$ is trivial will show that G is maximal. By Lemma 5.29 above, it suffices to show that each Lie algebra has a complete flag of characteristic ideals. They are as follows.

- If $\mathfrak{g} = \mathbb{R}^4 \rtimes \mathbb{R}[x^4]$:

$$\begin{aligned} \text{span}(x_1) &= \mathfrak{g}^4 \\ \text{span}(x_2, x_1) &= \mathfrak{g}^3 \\ \text{span}(x_3, x_2, x_1) &= \mathfrak{g}^2 \\ \text{span}(x_4, x_3, x_2, x_1) &= \{x \in \mathfrak{g} \mid \dim[x, \mathfrak{g}] \leq 1\} \end{aligned}$$

- If $\mathfrak{g} = \mathbb{R}^4 \rtimes \mathbb{R}[x^3, x]$:

$$\begin{aligned} \text{span}(x_1) &= \mathfrak{g}^3 \\ \text{span}(x_2, x_1) &= \mathfrak{g}^2 \\ \text{span}(x_4, x_2, x_1) &= \mathfrak{g}^2 + Z(\mathfrak{g}) \\ \text{span}(x_3, x_4, x_2, x_1) &= \{x \in \mathfrak{g} \mid \dim[x, \mathfrak{g}] \leq 1\} \end{aligned}$$

- If $\mathfrak{g} = \mathfrak{n}_4 \rtimes \mathbb{R}$ and $[x_5, x_4] = 0$:

$$\begin{aligned} \text{span}(x_1) &= \mathfrak{g}^3 \\ \text{span}(x_2, x_1) &= \mathfrak{g}^2 \\ \text{span}(x_5, x_2, x_1) &= \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] \subseteq \text{span}(x_1)\} \\ \text{span}(x_3, x_5, x_2, x_1) &= \{x \in \mathfrak{g} \mid [x, \text{span}(x_2, x_1)] = [x, \text{span}(x_5, x_2, x_1)]\} \end{aligned}$$

- If $\mathfrak{g} = \mathfrak{n}_4 \rtimes \mathbb{R}$ and $[x_5, x_4] = x_3$:

$$\begin{aligned} \text{span}(x_1) &= \mathfrak{g}^4 \\ \text{span}(x_2, x_1) &= \mathfrak{g}^3 \\ \text{span}(x_3, x_2, x_1) &= \mathfrak{g}^2 \\ \text{span}(x_5, x_3, x_2, x_1) &= \{x \in \mathfrak{g} \mid \dim[x, \mathfrak{g}] \leq 2\} \end{aligned} \quad \square$$

Case 2: G is not nilpotent. The descriptions of these Lie algebras are explicit enough that the eigenvalues can be verified to be real by inspection. So again it suffices to show that in each case, $\text{Aut } G = \text{Aut } \mathfrak{g}$ contains no nontrivial connected compact subgroups. By Lemma 5.29, it suffices to show that $(\text{Aut } \mathfrak{g})^0$ is upper-triangular in some basis.

In each case, an automorphism of \mathfrak{g} preserves the nilradical \mathfrak{n} and the decomposition of \mathfrak{n} into generalized eigenspaces of $\mathfrak{g}/\mathfrak{n}$, up to any reordering and scaling. Each generalized eigenspace has a natural flag—the filtration by the rank of generalized eigenvectors—which is also preserved. Except in two cases, this is enough data to make $(\text{Aut } \mathfrak{g})^0$ upper-triangular. The two cases and their additional data are as follows.

- In the $(\mathbb{R} \oplus \mathfrak{n}_3) \rtimes \mathbb{R}$ geometries, the flag $0 \subset \text{span}(x_1) \subset \text{span}(y, x_1)$ in the 0-eigenspace is preserved since $\text{span}(x_1) = [\mathfrak{n}, \mathfrak{n}]$.
- In $\mathbb{R}^3 \rtimes \mathbb{R}^2$, the subset of $\mathbb{R}^2 \cong \mathfrak{g}/\mathfrak{n}$ consisting of points that act with a zero eigenvalue is preserved. This is a set of three concurrent lines.

Proposition 5.31 (Claim of correctness of Figure 1.3). *All of the maximal geometries named above are distinct, with the exception that some of the geometries $\mathbb{R}^4 \rtimes \mathbb{R}$ with 4 distinct real eigenvalues may coincide with each other.*

Proof. Isomorphic geometries have isomorphic transformation groups; so it suffices to show that the corresponding Lie algebras are mutually non-isomorphic.

Referring to the calculation in Prop. 5.30, the nilpotent algebras are distinguished by whether they admit 4-dimensional abelian ideals and the number of nonzero terms in their lower central series.

The non-nilpotent algebras can be subdivided according to the isomorphism type of their nilradicals.

- Only $\mathbb{R}^3 \rtimes \mathbb{R}^2$ has nilradical \mathbb{R}^3 .
- The two $(\mathbb{R} \oplus \mathfrak{n}_3) \rtimes \mathbb{R}$ geometries are distinguished from each other by the dimensions of their centers—which are $\text{span}(x_1)$ if $[z, y] = x_1$, and $\text{span}(x_1, y)$ if $[z, y] = 0$.
- The non-nilpotent algebras $\mathfrak{g} = \mathbb{R}^4 \rtimes \mathbb{R}$ can be distinguished from each other by the Jordan blocks by which $\mathbb{R} \cong \mathfrak{g}/\mathbb{R}^4$ acts on the nilradical \mathbb{R}^4 —up to a scale factor, to account for the ability to rescale \mathbb{R} . This distinguishes all but the case when one action of \mathbb{R} on \mathbb{R}^4 has 4 distinct real eigenvalues that are a constant multiple of those of another action. \square

6 References

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